Definition :- (**Cyclic Module**) **المقاس الدائري**

Let (M ,+) be any R-module, then M is called **cyclic module** if $\exists$ x $\in$ M. Such that $<x>=Rx=\{r.\ x :\ \forall r \in R\}=M$, and x is called **generators** of module M .

Example(1):-

Let (Z , +) be a Z – module , then **Z is a cyclic module** since , $\exists$ 1 $\in$ Z . s. t. , $<1>= Z . 1 =\{ r. 1 :\ \forall r \in Z \}=\{ r :\ \forall r \in Z \} =Z$

Example(2):-

Let (Z_e ,+)be a Z– module, then **Z_e is a cyclic module** , since $\exists$ 2 $\in$ Z_e s .t , $<2> = 2Z =\{ r. 2 :\ \forall r \in Z \}=\{2. r :\ \forall r \in Z \}=Z_e$.

Example(3):- Let (Z_6 , $+_6$) be a Z−module . Find the generators of module Z_6 .

Solution :- To find the generators of (Z_6 , $+_6$)

$\because\ Z_6 = \{ \bar{0}, \bar{1}, \bar{2} , \bar{3} , \bar{4} , \bar{5} \}$

$\therefore\ (\bar{1}) =\{\bar{0}, \bar{1} , \bar{2} , \bar{3}, \bar{4}, \bar{5} \} = Z_6$

$(\bar{2}) = \{\bar{0} , \bar{2} , \bar{4}\} \neq Z_6$.

$(\bar{3})= \{\bar{0} , \bar{3} \} \neq Z_6$.

$(\bar{4}) = \{\bar{0} , \bar{2} , \bar{4}\} \neq Z_6$.

$(\bar{5}) = \{\bar{0} , \bar{1}, \bar{2} , \bar{3} , \bar{4}, \bar{5}\} =Z_6$

Therefore, the module Z_6 generated by $\bar{1}, \bar{5}$.

Proposition (6-3):- Let (M ,+) be an R-module , then if x $\in$ M .Then

$<x>= Rx$ is a sub module of module M .

Definition :- (**Simple R-module**) حلقة المقاس البسيط

Let $(M,+)$ be an R-module , then M is called **simple R-module** if the only sub module of M is trivial sub module which are M and $\{0\}$.

Example(1) :-

The module $(Z_P, +_P)$ as $Z-$module ,where p is a prime number is a **simple $Z-$module**

Since , the only sub module of Z_p is trivial sub module which are : Z_p and $\{0\}$. for example

Z_2 as $Z-$module is a simple $Z-$module

Z_3 as $Z-$module is a simple $Z-$module

Z_5 as $Z-$module is a simple $Z-$module

$\vdots$

Z_p as $Z-$module is a simple $Z-$module . Where p is prime.

Example(2) :-

The module $(Z_6, +_6)$ as $Z-$module **is not simple $Z-$module** , since

the module Z_6 has proper sub module which are :

$((2), +_6)$

$((3), +_6)$.

Proposition (6-4):- Every simply R−module is cyclic module.

Proof:- Let (M, +) be a simply R −module

< T.P. M is a cyclic module >

Let x ∈ M s.t, x ≠ 0 , then by Proposition (5) we get

<x> is a sub module of module M .

∴ either , <x>=M or <x>={0}

If <x>=M ⟹ M is a cyclic module generated by x .

If <x>={0} C! (since x ≠ 0)

Therefore, M is a cyclic module. □

ملاحظه :- عكس القضية أعلاه ليس من الضروري أن يكون صحيح المثال التالي يبين ذلك

Example: Give an example of cyclic module but is not simple module

Solution :-

Let $(Z_6, +_6)$ be a Z-module , then Z_6 is a cyclic module generated by 1 , 5 but is not simple Z-module , since the module Z_6 has proper sub module which are :

$(2) = \{\bar{0}, \bar{2}, \bar{4}\}$ and $(3) = \{\bar{0}, \bar{3}\}$

Definition: - (**Quotient Module**) مقاس ألقسمه

Let M be an R-module and N is a sub module of M. Then the set

$\mathbf{M/N} = \{ \boldsymbol{x} + \mathbf{N} : \boldsymbol{x} \in \mathbf{M}\}$ is called **quotient set of module** such that .

$\forall\ x + N ,\ y + N \in M/N$

$(x + N) + (y + N) = (x + y) + N$, for all x, y ∈M.

And

$r.(x + N) = r.x + N \ ,\forall\ r \in R$

Theorem (6-5):- **(Quotient module**) مقاس ألقسمه

If (R ,+,.)is a ring with identity and let M be an R-module and N is a sub module of module M. Then $(M/N\,,+)$ is an R–module, which is called **Quotient module** .

Proposition (6-7):-

If (R ,+,.)is a ring with identity and M be an R-module, N is a sub-module of M and $(M/N\,,+)$ is R–module, then $\forall\, x+N\,,\;\; y+N \in M/N$

$(\mathbf{1})\, a+N=N$ iff $a\in N$.

$(\mathbf{2})\, a+N=b+N$ iff $b-a\in N$.

Module Homomorphism

Definition :- (**Module Homomorphism**) التشاكل المقاسات

Let M_1 and M_2 be two R-modules, then a function $f: M_1 \rightarrow M_2$ is called **module homomorphism** or **R-homomorphism** iff :

1- $f(x+y) = f(x) + f(y) \quad , \forall\, x, y \in M_1$
2- $f(r.x) = r.f(x) \quad , \forall r \in R$ and $\forall\, x \in M_1$

Example (1):- Let M_1 and M_2 be two R-modules, and let $g: M_1 \longrightarrow M_2$ be a function s.t. , $g(m) = 0 \;, \forall m \in M_1$. Show that g is module homomorphism .

Solution :-

(1) $\forall\; m, n \in Z \Longrightarrow$ **To show ,** $\mathbf{g(m+n) = g(m) + g(n)}$

$\because$ $\boxed{g(m) = 0}$ and $\boxed{g(n) = 0}$, then

$\Rightarrow g(m+n) = 0 = 0 + 0 = g(m) + g(n)$.

(2) $\forall r \in Z \;, \forall\; m \in Z \Longrightarrow$ **To show ,** $\mathbf{g(r.m) = r.g(m)}$

$\Longrightarrow g(r.m) = 0$

$= r.0$

$= r.g(m)$

$\therefore$ **g is module homomorphism** .

Example(2):- Let $(Z, +)$ be a $Z-$module and $f: Z \longrightarrow Z$ be a function such that $f(n) = 2.n \;, \forall\, n \in Z$. Is f module homomorphism ?

Solution :-

(1) $\forall\ n, m \in Z$ $\Longrightarrow$ **To show, $f(n+m) = f(n) + f(m)$**

$\because$ $\boxed{f(n) = 2 . n}$ and $\boxed{f(m) = 2 . m}$, then

$\Rightarrow f(n+m) = 2 . (n+m) = 2 . n + 2 . m = f(n) + f(m).$

(2) $\forall r \in Z, \forall\ n \in Z$ $\Longrightarrow$ **To show, $f(r . n) = r . f(n)$**

$\Rightarrow\ f(r . n) = 2 . (r . n) = (2 . r) . n$

$= (r.2). n$

$= r . (2.n)$

$= r . f(n)$

$\therefore$ **f is module homomorphism .**

Example(3):- Let $(Z, +)$ be a Z_e -module and $f: Z \longrightarrow Z$ be a function s.t. $f(x) = 2x + 3$, $\forall x \in Z$, is f module homo ?

Solution :- $\forall\ x, y \in Z$, **To show $f(x+y) = f(x) + f(y)$,**

$\Longrightarrow$ $\boxed{f(x) = 2 . x + 3}$ and $\boxed{f(y) = 2 . y + 3}$, then

$\Rightarrow\ f(x+y) = 2 . (x+y) + 3 = (2 . x + 2 . y) + 3$ $\quad \cdots *$

and , $f(x) + f(y) = (2 . x + 3) + (2 . y + 3) = (2.x + 2.y) + 6$ $\quad \cdots **$

$\therefore$ by $*$ and $**$ we get , $f(x+y) \neq f(x) + f(y)$

$\therefore$ **f is not module homo.**

Proposition(6.8):-

Let M_1 and M_2 be two R-modules and $f: M_1 \rightarrow M_2$ be a module homomorphism then

1-$f(0_{M_1}) = 0_{M_2}$

2- $f(-x) = -f(x)$, $\forall x \in M_1$.

3- $f(x-y) = f(x) - f(y)$, $\forall x, y \in M_1$.

Theorem(6.9):-

If M_1 and M_2 are two R−modules , and if $f : M_1 \rightarrow M_2$ be a module homomorphism , then $f(M_1)$ is a sub module of module M_2 .

Proof :- $f(M_1)=\{ f(x) : x \in M_1\}$

< T.P. $f(M_1)$ is a sub module of M_2 >?

Since , M_1 is R-module

$\therefore 0_{M_1} \in M_1 \Longrightarrow f(0_{M_1}) \in f(M_1)$

$\Longrightarrow 0_{M_2} \in f(M_1)$ (**by proposition (9)** $f(0_{M_1}) = 0_{M_2}$)

$\Longrightarrow f(M_1) \neq \emptyset$

(i) Let $f(x),f(y) \in f(M_1)$,where $x , y \in M_1$, **<T. p , $f(x) + f(y) \in f(M_1)$>**

$\because M_1$ be a R-module and $x , y \in M_1$

$\Longrightarrow x + y \in M_1$

$\Longrightarrow f(x + y) \in f(M_1)$ (بأخذ f للطرفيين)

$\Longrightarrow f(x) + f(y) \in f(M_1)$ (since f is module homo)

(ii) Let $\forall\, r \in R$ and $\forall\, f(x) \in f(M_1)$, $x \in M_1$

<T.P, $r . f(x) \in f(M_1)$ >

$\because r \in R$ and $x \in M_1$ and M_1 is a R−module.

$\Longrightarrow r . x \in M_1$

$\Longrightarrow f(r . x) \in f(M_1)$ (بأخذ f للطرفيين)

$\Longrightarrow r . f(x) \in f(M_1)$ (since f is module homo.)

Therefore, by steps (i) and (ii) we get, $(f(M_1) , +)$ is a sub module of module $(M_2, +)$. □

Theorem (6-10):- If M_1 and M_2 are two R−modules , and $f : M_1 \to M_2$ a a module homomorphism and onto function . Then

(**1**) If (N ,+) is a sub module of module M_1 . Then (f(N) , +)is also sub module of module M_2 . (البرهان واجب)

(**2**) If (K ,+) is sub module of module M_2 . Then ($f^{-1}(K)$, +) is also sub module of module M_1 . (البرهان واجب)