10- $$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x+y)}{\cancel{(x-y)}} = \lim_{(x,y)\to(1,1)} (x+y) = (1+1) = 2$$

11- $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2+y^2}{x^2-y^2}\right] = \lim_{x\to 0}\left[\frac{x^2}{x^2}\right] = 1$$

and let $\lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2+y^2}{x^2-y^2}\right] = \lim_{y\to 0}\left[\frac{y^2}{-y^2}\right] = \lim_{y\to 0}(-1) = -1$

since $\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2+y^2}{x^2-y^2}\right] + \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2+y^2}{x^2-y^2}\right]$

$\therefore$ The Limit is not exist

Example :-

(6)

Does $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ exist or not?

sol :-

$$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^3+y^3}{x^2+y^2}\right] = \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^3+y^3}{x^2+y^2}\right] = \lim_{y\to 0} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0$$

Then

$$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^3+y^3}{x^2+y^2}\right] = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^3+y^3}{x^2+y^2}\right]$$

$\therefore$ The Limit exist

Example : show that the limit is not exist of the function $f(x,y) = \frac{x^2+y^2}{xy}$ along the line $y = mx$, $-\infty < m < \infty$

as $(x,y) \longrightarrow (0,0)$

solution :-

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2+m^2x^2}{x^2 m}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^2}(1+m^2)}{\cancel{x^2} m} = \lim_{(x,y)\to(0,0)} \frac{1+m^2}{m}$$

$$= \frac{1+m^2}{m}, \quad -\infty < m < \infty$$

The Limit is not exist, Because There is no single L

Example :- Examine the Limit of $f(x,y) = \frac{x^2y}{x^4+y^2}$
as $(x,y) \to (0,0)$ along
① The Line $y = mx$ ② along $y = x^2$. Does f have a limit as $(x,y) \to (0,0)$?
solution :-
① Along the line $y = mx$, $m \neq 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{x^2 \cdot mx}{x^4+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^2(x^2+m)} = \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2} = 0$$

② Along $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^2}} \frac{x^2x^2}{x^4+x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}$$

since

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{x^2y}{x^4+y^2} \neq \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^2}} \frac{x^2y}{x^4+y^2}$$

$\therefore$ f has no limit as $(x,y) \to (0,0)$

Example :-

show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+x^2+y^2}{x^2+y^2}$ is not exist

solution :-

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right]$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

and

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right]$$

$$= \lim_{y\to 0} \frac{y^2}{y^2} = 1$$

Although $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \dfrac{xy+x^2+y^2}{x^2+y^2}\right] = \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} \dfrac{xy+x^2+y^2}{x^2+y^2}\right]$

But, we want to show $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+x^2+y^2}{x^2+y^2}$ is not exist

we will take the limit on the line $y=2x$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2+x^2+(2x)^2}{x^2+(2x)^2}$$

$$= \lim_{x\to 0} \frac{7x^2}{5x^2} = \frac{7}{5}$$ which is not equal 1.

Exercise

1. Does $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \dfrac{x^4 y}{x^6+y^3}$ exist or not?

Remark :- For the limits of the functions of three or more variables we will apply theorems similar to theorem (1) and (2) page (4)

Examples :-

1. $\lim\limits_{(x,y,z)\to(2,4,3)} z = 3$

2. $\lim\limits_{(x,y,z)\to(2,3,3)} \dfrac{xy^2-xz^2}{y-z}$

$= \lim\limits_{(x,y,z)\to(2,3,3)} \dfrac{x(y^2-z^2)}{(y-z)} = \lim\limits_{(x,y,z)\to(2,3,3)} \dfrac{x\cancel{(y-z)}(y+z)}{\cancel{(y-z)}}$

$= 12$

3. $\lim\limits_{(x,y,z)\to(1,1,1)} \sqrt{x^2+y^2+z^2-1} = \sqrt{2}$

Continuity

Definition:- A function $f(x,y)$ is said to be continuous at the point (x_0, y_0), if

① f is defined at (x_0, y_0)

② $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and

③ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Remark:-

1) If $f(x,y)$, $g(x,y)$ are both continuous at apoint (x_0, y_0) then
the sum $(f+g)$, difference $(f-g)$, product $(f \cdot g)$ and quotient $\frac{f}{g}$, $g \neq 0$ are continuous at (x_0, y_0)

Ex let $f(x,y) = xy + x^2$ and $g(x,y) = xy^2$ which are continuous at the point $(1,2)$, then the functions
$(f+g)(x,y) = xy + x^2 + xy^2$,
$(f-g)(x,y) = xy + x^2 - xy^2$,

$(f \cdot g)_{(x,y)} = (xy + x^2)(xy^2)$, and

$\frac{f}{g}(x,y) = \frac{xy + x^2}{xy^2}$

are all continuous at the point $(1,2)$

2. Polynomial function of two varibles are continuous everywhere

Ex $f(x,y) = x^6 + x^2y^2 + y^3 + 50$ is continuous everywhere.

3. Rational function of two Varible are continuous wherever the denominator is not zero.

Ex $f(x,y) = \dfrac{2x+4y}{y^2-4x}$ is continuous every where except at poin $y^2 = 4x$.

4. IF $f(x,y)$, $g(x,y)$ are continuous, then $f \circ g$ are continuous.
and $g \circ f = g(f(x))$

i.e IF $z = f(x,y)$ is continuous function of x, y and $w = g(z)$ is a continuous function of z, then the composite $w = g(f(x,y))$ is continuous.

Ex, Let $f(x,y) = x-y$, $g(z) = e^{z}$, $z = (x,y)$

$\therefore\ g \circ f = g(f(x,y)) = g(x-y) = e^{x-y}$.

Ex, Show that the Function $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x+y} & (x,y) \neq (4,2) \\ 2 & (x,y) = (4,2) \end{cases}$
is continuous at the point $(4,2)$.

Sol,

1. $f(x,y) = f(4,2) = 2$ is defined

2. $\lim\limits_{(x,y)\to(4,2)} f(x,y) = \lim\limits_{(x,y)\to(4,2)} \dfrac{x^2-y^2}{x+y} = \dfrac{16-4}{6} = 2$.

3. $\lim\limits_{(x,y)\to(4,2)} f(x,y) = f(4,2) = 2$

$\therefore$ f is continuous at the point $(4,2)$