The second partial derivative of f :-

IF f is a function of two varibles x and y, the second partial derivative of f are denoted as follows :-

1. $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

2. $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

3. $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ Diff first w.r.t x, then w.r.t y

4. $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ Diff first w.r.t y, then w.r.t x

Def: A Function $z = f(x,y)$ is called harmonic if f satisfy Laplaces equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Ex, Determine whether the Following Functions is harmonic or not

1) $f(x,y) = e^{-y} \cos x$.

sol, $f_x = -e^{-y} \sin x \Rightarrow f_{xx} = -e^{-y} \cos x$

$f_y = -e^{-y} \cos x \Rightarrow f_{yy} = e^{-y} \cos x$

we get $f_{xx} + f_{yy} = -e^{-y} \cos x + e^{-y} \cos x = 0$

$\therefore$ f is harmonic Function.

2. $Z = \sin^{-1}\left(\frac{y}{x}\right)$ (Exc).

3. $f(x,y) = \ln (x^2+y^2)^2$ (Exc).

Definition: A function $z = f(x,y)$ (of two independent variables x and y) is called harmonic if f satisfy Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Example: Determine whether the function $f(x,y) = \sin^{-1}\left(\frac{y}{x}\right)$ where $\left|\frac{y}{x}\right| < 1$, is harmonic or not.

solution:

$$f_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2\sqrt{1-\frac{y^2}{x^2}}}$$

$$= \frac{-y}{\sqrt{x^4-x^2y^2}} = -y\left(x^4-x^2y^2\right)^{-\frac{1}{2}}.$$

$$f_{xx} = \frac{1}{2}y\left(x^4-x^2y^2\right)^{-\frac{3}{2}} \cdot \left(4x^3-2xy^2\right)$$

$$f_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{1}{x} = \frac{1}{x\sqrt{1-\frac{y^2}{x^2}}} = \frac{1}{\sqrt{x^2-y^2}}$$

$$= \left(x^2-y^2\right)^{-\frac{1}{2}}$$

$$f_{yy} = -\frac{1}{2}\left(x^2-y^2\right)^{-\frac{3}{2}}(-2y) = y\left(x^2-y^2\right)^{-\frac{3}{2}}$$

$$f_{xx} + f_{yy} = \frac{1}{2}y\left(x^4-x^2y^2\right)^{-\frac{3}{2}}\left(4x^3-2xy^2\right) + y\left(x^2-y^2\right)^{-\frac{3}{2}}$$

$$\neq 0$$

$\therefore$ f is not harmonic.

Example: Determin whether the following function is harmonic or not.

① $f(x,y) = \ln (x^2+y^2)^2$

solution:

$$f_x = \frac{2(x^2+y^2)\cdot 2x}{(x^2+y^2)^2} = \frac{4x}{x^2+y^2}$$

$$f_{xx} = \frac{4(x^2+y^2) - 4x\cdot 2x}{(x^2+y^2)^2} = \frac{4y^2-4x^2}{(x^2+y^2)^2}$$

$$f_y = \frac{2(x^2+y^2)\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{4y}{x^2+y^2}$$

$$f_{yy} = \frac{4(x^2+y^2) - 4y\cdot 2y}{(x^2+y^2)^2} = \frac{4x^2+4y^2-8y^2}{(x^2+y^2)^2}$$

$$= \frac{4x^2-4y^2}{(x^2+y^2)^2}$$

$$f_{xx} + f_{yy} = \frac{4y^2-4x^2}{(x^2+y^2)^2} + \frac{4x^2-4y^2}{(x^2+y^2)^2}$$

$$= 0$$

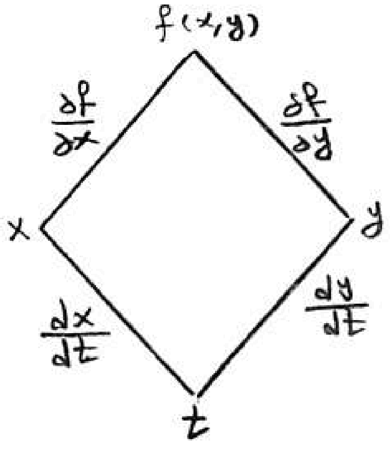
$\therefore$ f is harmonic function

② $z = f(x,y) = \cos^{-1}\left(\frac{y}{x}\right)$ where $\left|\frac{y}{x}\right| < 1$ (H.w.)

Chain Rule :-

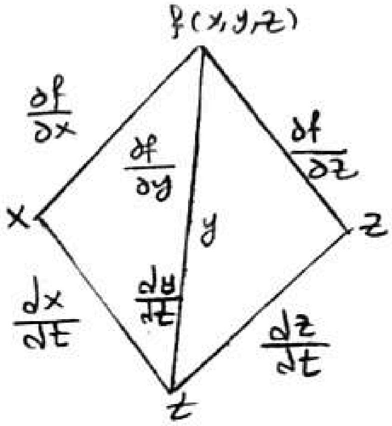
1- if $w = f(x,y)$ has continuous partial derivative f_x and f_y and if $x = x(t)$, $y = y(t)$ are diff. function of t $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ then the composite $w = f(x(t), y(t))$ is differentiable of t and

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$



2. If $w = f(x,y,z)$ has continuous partial derivative and $x = x(t)$, $y = y(t)$, $z = z(t)$ [x, y, z are function of t], z is differentiable function of t, then the appropriate formulas for $\frac{df}{dt}$ are the Following

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}$$



Ex1 Use the chain rule to find the derivative of $f(x,y) = x^5 + 6y^2$ w.r.t t along the path $x = \ln(t)$, $y = e^t$

Sol

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= 5x^4 \cdot \frac{1}{t} + 12y \cdot e^t = 5(\ln(t))^4 \cdot \frac{1}{t} + 12(e^t) \cdot e^t$$

Ex2 Use the chain rule to find the derivative of $w = x^3 - 3y + 5z$, $x = t^2$, $y = e^t$, $z = \cos t$.

Sol

$$\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt} + w_z \cdot \frac{dz}{dt}$$

$$= 3x^2 \cdot (2t) + (-3) \cdot e^t + 5(-\sin t)$$

$$= 6t(t^2)^2 - 3e^t - 5\sin t = 6t^5 - 3e^t - 5\sin t$$

Another Method To check our answer

Ex3, use the chain rule to find the derivative of $z = \sin y + x e^{x}$, $x = \ln(2t+10)$, $y = 5t+12$.

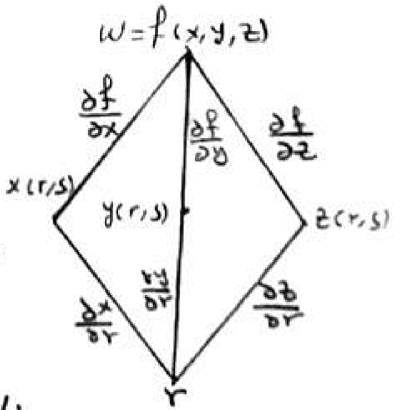
Sol, Exc.

chain rule for Function defined on surface :-

If $w = f(x,y,z)$ and x, y, z are a function of r, s.
i.e $x = x(r,s)$, $y = y(r,s)$, $z = z(r,s)$ and x, y, z, f have continuous partial derivative, then $w = f(x(r,s), y(r,s), z(r,s))$ is a function of r, s and the partial derivative of w w.r.t r and s exist, and given by the Following equation :-

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$



Ex2 Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ of the Following Function
$w = f(x,y,z) = x^2 + 5xy^2 - zy$, $x = 2r + s^2$, $y = s + \ln(r)$
$z = e^{r+s}$

1. $$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= (2x + 5y^2)\cdot 2 + (10xy - z)\frac{1}{r} + (-y)\cdot e^{r+s}$$ $\quad -e^{r+s})\frac{1}{r} \quad e^{r+s}$

$$= 2\left[2(2r+s^2) + 5(s+\ln(r))^2\right] + (10(2r+s^2)(s+\ln(r)) - (s+\ln(r))$$