Gradient vector

Def:- IF the partial derivative of $f(x,y,z)$ are defined at at $P_0(x_0,y_0,z_0)$ then the gradient of f at P_0 is the vector

$$\vec{\nabla f} = \frac{\partial f}{\partial x} i \Big|_{P_0} + \frac{\partial f}{\partial y} j \Big|_{P_0} + \frac{\partial f}{\partial z} k \Big|_{P_0}$$

Directional Derivative:-

Def:- IF $f(x,y,z)$ has continuous partial derivative at $P_0(x_0,y_0,z_0)$ and u is a unit vector, then the derivative of f at P_0 in the direction of u is the number.

$$(D_{\vec{u}} f)_{P_0} = (\vec{\nabla f})_{P_0} \cdot \vec{u}$$

which is the scaler product of u and the gradient of f at P_0

Note:- 1. Another notation is use for the gradient of f is grad f.

2. The symbol ∇f may be read "grad f" or "gradient f,

$$\vec{u} = \frac{u}{\|u\|}$$

Ex: Find the directional derivative of $f(x,y) = x^2 + xy + y^2$ at $P_0(1,2)$ in the direction of $\vec{A} = 2i + 3j$.

Sol: $(D_{\vec{u}} f)_{P_0} = (\vec{\nabla} f)_{P_0} \cdot \vec{u}$.

$$\vec{\nabla} f = f_x\big|_{P_0} i + f_y\big|_{P_0} j$$

$$\Rightarrow f_x = 2x + y \Rightarrow f_x\big|_{P_0} = 4$$

$$f_y = x + 2y \rightarrow f_y\big|_{P_0} = 5.$$

$$\vec{\nabla} f = 4i + 5j$$

$|\vec{A}| = \sqrt{4+9} = \sqrt{13}$ is not unit vector.

$$\therefore \vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2i+3j}{\sqrt{13}} = \frac{2}{\sqrt{13}} i + \frac{3}{\sqrt{13}} j$$

$$\therefore (D_{\vec{u}} f)_{P_0} = (4i + 5j)\left(\frac{2i}{\sqrt{13}} + \frac{3}{\sqrt{13}} j\right) = \frac{8}{\sqrt{13}} + \frac{15}{\sqrt{13}} = \frac{23}{\sqrt{13}}$$

Ex2 Find the directional derivative of $f(x,y,z) = e^{x}\cos(y+z)$ at the point $P_0(0,0,0)$ in the direction $\vec{A} = 2i + j - 2K$.

Sol $\vec{\nabla} f = f_x\big|_{P_0} i + f_y\big|_{P_0} j + f_z\big|_{P_0} K$.

$$f_x = e^{x}\cos(y+z) \rightarrow f_x\big|_{P_0} = 1.$$

$$f_y = -e^{x}\sin(y+z) \rightarrow f_y\big|_{P_0} = 0$$

$$f_z = -e^{-x}\sin(y+z) \rightarrow f_z\big|_{P_0} = 0$$

$\therefore \vec{\nabla f} = i + 0\cdot j + 0\cdot K = i$

$|\vec{A}| = \sqrt{4+1+4} = \sqrt{9} = 3 \neq 1$ is not unit vector.

$$\therefore \vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2i+j-2K}{3}$$

$$= \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}K.$$

$$\therefore (D_{\vec{u}}f)_{p_0} = \vec{\nabla f}\cdot\vec{u} = i\cdot\left(\frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}K\right) = \frac{2}{3}.$$

ملاحظة ①: - عند كتابة الاشتقاق الاتجاهي للدالة بدلالة زاوية اتجاهها :-

$$(D_{\vec{u}}f)_{p_0} = \left(f_x\big|_{p_0} i + f_y\big|_{p_0} j\right)\cdot\left(i\cos\theta + j\sin\theta\right).$$

$$= f_x\big|_{p_0}\cos\theta + f_y\big|_{p_0}\sin\theta.$$

Ex) Find the direction derivative of $f(x,y) = x^2 - 3xy + 2y^2$ at $p_0(1,2)$ in the direction 1. $\theta = \frac{\pi}{6}$ 2. $\theta = \frac{\pi}{4}$ ③ $\theta = 2\pi$

Sol) 1. $(D_u f)_{p_0} = f_x\big|_{p_0}\cos\theta + f_y\big|_{p_0}\sin\theta.$

$\Rightarrow f_x = 2x - 3y \Rightarrow f_x\big|_{p_0} = -8$

$f_y = -3x + 4y \Rightarrow f_y\big|_{p_0} = 11.$

$$\therefore (D_u f)_{p_0} = -8\cos\left(\frac{\pi}{6}\right) + 11\sin\left(\frac{\pi}{6}\right) \Rightarrow -8\cdot\frac{\sqrt{3}}{2} + 11\cdot\frac{1}{2}$$

$$= \frac{-8\sqrt{3}+11}{2}$$

2. Exe 3. Exe.

ملاحظة ② :- اذا كانت f دالة لثلاث متغيرات $f(x,y,z)$ بحيث ان لها اشتقاقات جزئية f_x, f_y, f_z ثم المشتقة الاتجاهية في النقطة $P_0(x_0,y_0,z_0)$

$$(1)\; D_{\vec{u}}f)_{P_0} = f_x\Big|_{P_0}\cos\alpha + f_y\Big|_{P_0}\cos\beta + f_z\Big|_{P_0}\cos\gamma$$

حيث L مستقيم يمر بـ P_0 ويصنع مع المحاور x, y, z الزوايا α, β, α على الترتيب

Ex, Estimate how much $f(x,y,z) = xe^{y} + yz$ will change (Δf) if the point $p(x,y,z)$ is moved from $p_0(2,0,0)$ straight toward $p_1(4,1,-2)$ a distance of $\Delta s = 0.1$ units

Sol, To find the vector $\Rightarrow \overrightarrow{P_0P_1} = (4-2)i + (1-0)j + (-2-0)K$

$\therefore \overrightarrow{P_0P_1} = 2i + j - 2K$. and $|\overrightarrow{P_0P_1}| = \sqrt{9} = 3$ is not unit vector

$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2i + j - 2K}{3} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}K.$$

$$\vec{\nabla} f = f_x\Big|_{P_0} i + f_y\Big|_{P_0} j + f_z\Big|_{P_0} K$$

$\Rightarrow f_x = e^{y} \Rightarrow f_x\Big|_{P_0} = 1$

$f_y = xe^{y} + z \Rightarrow f_y\Big|_{P_0} = 2$

$f_z = y \rightarrow f_z\Big|_{P_0} = 0$

The derivative of f at P_0 in the direction $\overrightarrow{P_0P_1}$ is

$$(D_u f)_{P_0} = \vec{\nabla f} \cdot \vec{u}$$

$$= (i + 2j) \cdot \left(\frac{2}{3} i + \frac{1}{3} j - \frac{2}{3} K\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

$$\therefore \Delta f \simeq (D_{\hat{u}} f)_{P_0} \cdot \Delta S$$

$$\simeq \frac{4}{3} \cdot (0.1) \simeq 0.13.$$

Exc, Find $(D_u f)_{P_0}$ of the Following Functions.

1. $f(x,y) = e^x \sin \pi y$ $\quad P_0(1,0)$, $\vec{A} = 4i + 4j$

2. $f(x,y,z) = e^{3x+4y} \cos 5z$, $P_0(0,0,\frac{\pi}{6})$, $\vec{B} = i + j - K$.

3. $f(x,y) = \dfrac{x - y^2}{x}$, $P_0(1,1)$, $\vec{C} = 12i - j$

4. $f(x,y) = \ln(x^2 + y^2 + z^2)$, $P_0(2,1,0)$, $\vec{D} = (2i - j - K)$

5. $f(x,y,z) = e^x \cos yz$, $P_0(0,0,0)$, $\vec{E} = 2i + j - 2K$.