ملاحظة ② :- اذا كانت f دالة لثلاث متغيرات $f(x,y,z)$
بحيث ان لها المشتقات الجزئية f_x, f_y, f_z فالمشتقة
الاتجاهية في النقطة $P_0(x_0, y_0, z_0)$

$$\vec{u} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$

$$(D_{\vec{u}} f)_{P_0} = f_x\Big|_{P_0} \cos\alpha + f_y\Big|_{P_0} \cos\beta + f_z\Big|_{P_0} \cos\gamma$$

حيث L مستقيم يمر بـ P_0 ويصنع مع المحاور x, y, z الزوايا α, β, γ
على الترتيب

Example :- Find the directional derivative of $f(x,y,z) = 2xy + 3yz$ at the point $P_0(2,4,-1)$ in the direction $\vec{A} = 2i + 6j - 3K$

Solution :-

$$(D_{\vec{u}} f)_{P_0} = \vec{\nabla} f \big|_{P_0} \cdot \vec{u}$$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2i + 6j - 3K}{\sqrt{4+36+9}} = \frac{2}{7} i + \frac{6}{7} j - \frac{3}{7} K$$

$$= \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$

$$\vec{\nabla f} = f_x \big|_{P_0} i + f_y \big|_{P_0} j + f_z \big|_{P_0} K$$

$$f_x = 2y \Rightarrow f_x(2,4,-1) = 8$$
$$f_y = 2x + 3z \Rightarrow f_y(2,4,-1) = 1$$
$$f_z = 3y \Rightarrow f_z(2,4,-1) = 12$$

$$(D_{\vec{u}} f)_{P_0} = f_x \big|_{P_0} \cos\alpha + f_y \big|_{P_0} \cos\beta + f_z \big|_{P_0} \cos\gamma$$

$$= 8\left(\frac{2}{7}\right) + 1\left(\frac{6}{7}\right) + 12\left(-\frac{3}{7}\right)$$

$$= \frac{-14}{7} = -2$$

Exc, Find $(D_u f)_{P_0}$ of the Following Functions.

1. $f(x,y) = e^{x} \sin \pi y \qquad P_0(1,0) \;,\; \vec{A} = 4i + 4j$

2. $f(x,y,z) = e^{3x+4y} \cos 5z \;,\; P_0(0,0,\frac{\pi}{8}) \;,\; \vec{B} = i + j - k.$

3. $f(x,y) = \dfrac{x - y^2}{x} \;,\; P_0(1,1) \;,\; \vec{C} = 12i - j$

4. $f(x,y) = \ln(x^2 + y^2 + z^2) \;,\; P_0(2,1,0) \;,\; \vec{D} = (2i - j - k)$

5. $f(x,y,z) = e^{x} \cos yz \;,\; P_0(0,0,0) \;,\; \vec{E} = 2i + j - 2k.$

Properties of the directional derivative :-

$$(D_{\vec{u}}f)_{P_0} = \nabla f|_{P_0} \cdot \vec{u} = |\nabla f| \cos\theta .$$

1. The directional derivative has its Largest positive value when $\cos\theta = 1$ or when u is the direction of the gradient, that is, f increases most rapidly in its domain in the direction of $\vec{\nabla f}$

∴ The derivative in this direction is

$$(D_{\vec{u}}f)_{P_0} = |\vec{\nabla f}| . \cos(0) = |\vec{\nabla f}|$$

المشتقة الاتجاهية تأخذ اعلى قيمة موجبة عندما $\cos\theta = 1$ او عندما تكون $\vec{u}$ باتجاه انحدار الدالة $\vec{\nabla f}$ عندها تتزايد الدالة بسرعة في مجال $\vec{\nabla f}$.
اي بمعنى ان الدالة تتغير بأقصى سرعتها في الاتجاه المعطى بالمتجه $\vec{\nabla f}$ نفسه
زيادة على ذلك فان المشتقة الاتجاهية في الاتجاه الموجب مساوية للقيمة المطلقة للانحدار .

2. Similary f decreases most rapidly in the direction of $-\vec{\nabla f}$
The derivative in the direction is

$$(D_{\vec{u}}f)_{P_0} = |\vec{\nabla f}| \cos\pi = -|\vec{\nabla f}| .$$

3. Any direction $\vec{u}$ perpendicular to the gradient ∇f is a direction of zero

$$(D_u f) = |\vec{\nabla f}| \cos\frac{\pi}{2} = |\nabla f| . 0 = 0 .$$

Ex a. Find the derivative of $f(x,y) = 100 - x^2 - y^2$ at the point $P_0(3,4)$ in the direction of the unit vector $\vec{u} = u_1 i + u_2 j$

b. In what direction in its domain (the xy-plane) is f increasing most rapidly at P_0?

c. what the derivative of f in this direction?

d. Identify the direction in which the derivative of f is zero?

Sol: (a) $f(x,y) = 100 - x^2 - y^2 \Rightarrow f_x|_{P_0} = -2x|_{P_0} = -6$, $f_y|_{P_0} = -2y|_{P_0} = -8$

$\therefore (D_u f)_{P_0} = \nabla f|_{P_0} \cdot \vec{u} =$

$= (-6i - 8j)(u_1 i + u_2 j) = -6u_1 - 8u_2.$

(b) The Function increases most rapidly in the direction of the gradient.

$\nabla f = -6i - 8j \Rightarrow |\nabla f| = \sqrt{36+64} = \sqrt{100} = \sqrt{10}.$

$\therefore$ The direction of the gradient is :-

$$\vec{u} = \frac{\nabla f|_{P_0}}{|\nabla f|} = \frac{-6i - 8j}{10} = \frac{-3}{5} i - \frac{4}{5} j.$$

(c) The derivative in this direction is $(D_u f)_{P_0} = |\nabla f| \cos 0 = |\nabla f| = 10$

(d) The derivative of f is zero in the directions perpendicular to ∇f, we can obtain one of these directions by the interchanging the components of $\vec{u} = \frac{-3}{5} i - \frac{4}{5} j$ and chaining the sign of new first components.

$\therefore$ The result is $n = \frac{4}{5} i - \frac{3}{5} j$ or $-n = -\frac{4}{5} i + \frac{3}{5} j$

This unit vector which is normal on u.

$\therefore (D_u f)_{P_0} = \nabla f \cdot n = (-\frac{3}{5} i - \frac{4}{5} j)(\frac{4}{5} i - \frac{3}{5} j) = 0.$