2) By using the bisection method, find the approximate root for the following equations:

i) $\cos x = x^2$ in the interval $[0, 1]$ and $\epsilon = 0.01$.

ii) $\frac{1}{x} + 1 = 0$ in the interval $[-2, -0.5]$ and $\epsilon = 0.05$.

4) Newton-Raphson method:

Let x_0 selected to be an approximate root of the equation $f(x)=0$. If the exact value of the root is $x_1 = x_0 + h$, then $f(x_1)=0$.

Therefore by Taylor's series we have

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since we take h to be small, then we can neglecting h^2 and h^3 and higher powers of h, we get that

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Thus $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is a closer approximation to the exact value of the root.

Similarly, if we let x_1 to be an approximate root of the equation $f(x)=0$, then a better approximation will be x_2 where

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and so on we have that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ---(*)$$

The formula $(*)$ is known as the Newton - Raphson formula or Newton's iteration formula.

Example: Find a positive root of the equation $x^4 = x + 10$ correct to three decimal places, using Newton-Raphson method.

Solution:

The given equation can be written as
$x^4 - x - 10 = 0$

Let $f(x) = x^4 - x - 10$, then $f'(x) = 4x^3 - 1$.

Since $f(1) = 1^4 - 1 - 10 = -10$ ($-ve$) and $f(2) = 2^4 - 2 - 10 = 16 - 2 - 10 = 4$ ($+ve$), then a root of $f(x)=0$ lies between 1 and 2.

Let us take $x_0 = 2$.

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For $n=0$, we get the first approximation x_1 of the root which is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{4(2^3)-1} = 2 - \frac{4}{31}$$

$= 1.870967742$.

For $n=1$, we get the second approximation x_2 of the root which is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.870967742 - \frac{f(1.870967742)}{f'(1.870967742)}$$

$$= 1.870967742 - \frac{(1.870967742)^4 - 1.870967742 - 10}{4(1.870967742)^3 - 1}$$

$$= 1.870967742 - \frac{0.382674569}{25.19744218} = 1.855780702$$

For $n=2$, we get the third approximation x_3 of the root which is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.855780702 - \frac{f(1.855780702)}{f'(1.855780702)}$$

$$= 1.855780702 - \frac{(1.855780702)^4 - 1.855780702 - 10}{4(1.855780702)^3 - 1}$$

$$= 1.855780702 - \frac{0.0048181359}{24.56465605} = 1.855584561$$

Since $x_2 = 1.856$ (rounded to three decimal places) and $x_3 = 1.856$ (rounded to three decimal places) (i.e. $x_2 = x_3$). Therefore the root $\bar{x}$ is 1.856 correct to three decimal places.

Example: Find a positive root of the equation $x^3-x-1=0$ correct to five decimal places, using Newton–Raphson method.

Solution:

Let $f(x)=x^3-x-1$, then $f'(x)=3x^2-1$.

Since $f(1)=1^3-1-1=-1$ $(-ve)$ and $f(2)=2^3-2-1=5$ $(+ve)$, then a root of $f(x)=0$ lies between 1 and 2.

Let us take $x_0=1$

Newton–Raphson formula is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

For $n=0$, we get the first approximation x_1 of the root which is

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{1^3-1-1}{3(1^2)-1}=1+\frac{1}{2}=1.5$$

For $n=1$, we get the second approximation x_2 of the root which is

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=1.5-\frac{(1.5)^3-1.5-1}{3(1.5)^2-1}$$

$$=1.5-\frac{3.375-1.5-1}{6.75-1}=1.5-\frac{0.875}{5.75}$$

$$=1.347826087.$$