Partial Derivative

Let $z = f(x,y)$ be a Function of two independent varibles x and y

1. IF y is Fixed, then f will be a Function of one varible, then we can derive with respect to (w.r.t) x. this derivative is called partial derivative of f w.r.t x and denoted by f_x or $\frac{\partial f}{\partial x}$, hence f_x is a Function and its value at (x_0, y_0) =

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

2. IF x is fixed, then will be a Function of one varible y, then we derive w.r.t y, this derivative is called partial derivative of f w.r.t y and denoted by f_y or $\frac{\partial f}{\partial y}$ hence f_y is a Function and its value at (x_0, y_0) =

$$\frac{\partial f}{\partial y}(x_0, y_0) \text{ or } f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Ex: Find $f_x(x,y)$ and $f_y(x,y)$ by using the limit definition of partial derivative

① $f(x,y) = 3x + 2y$ ② $f(x,y) = \sqrt{xy}$

solution:

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x+\Delta x) + 2y - (3x + 2y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x} + 3\Delta x \cancel{+2y} \cancel{-3x} \cancel{-2y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\cancel{\Delta x}}{\cancel{\Delta x}} = \boxed{3}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3x + 2(y+\Delta y) - (3x+2y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\cancel{3x} \cancel{+2y} + 2\Delta y \cancel{-3x} \cancel{-2y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2\cancel{\Delta y}}{\cancel{\Delta y}} = \boxed{2}$$

(2) $f(x,y) = \sqrt{xy}$

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)y} - \sqrt{xy}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{xy + \Delta x y} - \sqrt{xy}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{xy+\Delta xy} - \sqrt{xy}}{\Delta x} \times \frac{\sqrt{xy+\Delta xy} + \sqrt{xy}}{\sqrt{xy+\Delta xy} + \sqrt{xy}}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{xy} + \Delta xy - \cancel{xy}}{\Delta x(\sqrt{xy+\Delta xy} + \sqrt{xy})} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x} y}{\cancel{\Delta x}(\sqrt{xy+\Delta xy} + \sqrt{xy})}$$

$$= \lim_{\Delta x \to 0} \frac{y}{\sqrt{xy+\Delta xy} + \sqrt{xy}} = \frac{y}{\sqrt{xy + y(0)} + \sqrt{xy}}$$

$$= \boxed{\frac{y}{2\sqrt{xy}}}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\sqrt{x(y+\Delta y)} - \sqrt{xy}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\sqrt{xy + x\Delta y} - \sqrt{xy}}{\Delta y} \times \frac{\sqrt{xy+x\Delta y} + \sqrt{xy}}{\sqrt{xy+x\Delta y} + \sqrt{xy}}$$

$$= \lim_{\Delta y \to 0} \frac{\cancel{xy} + x\Delta y - \cancel{xy}}{\Delta y(\sqrt{xy+x\Delta y} + \sqrt{xy})} = \lim_{\Delta y \to 0} \frac{x\cancel{\Delta y}}{\cancel{\Delta y}(\sqrt{xy+x\Delta y} + \sqrt{xy})}$$

$$= \lim_{\Delta y \to 0} \frac{x}{\sqrt{xy+x\Delta y} + \sqrt{xy}} = \frac{x}{\sqrt{xy+(0)} + \sqrt{xy}}$$

$$= \boxed{\frac{x}{2\sqrt{xy}}}$$

Ex Find $f_x(x,y)$ by using the limit definition

1) $f(x,y) = \cos xy$

Solution

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos((x+\Delta x)y) - \cos xy}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(xy + y\Delta x) - \cos xy}{\Delta x}$$

$\because \boxed{\cos(a+b) = \cos a \cos b - \sin a \sin b}$ قانون

$$f_x = \lim_{\Delta x \to 0} \frac{\cos xy \cos y\Delta x - \sin xy \sin y\Delta x - \cos xy}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos xy \cos y\Delta x - \cos xy - \sin xy \sin y\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos xy \frac{[\cos y\Delta x - 1]}{\Delta x} \times \left(\frac{y}{y}\right)$$

$$- \lim_{\Delta x \to 0} \sin xy \frac{\sin y\Delta x}{\Delta x} \times \left(\frac{y}{y}\right)$$

$$= \lim_{\Delta x \to 0} y \cos x\,(0) - \lim_{\Delta x \to 0} y \sin xy \cdot (1)$$

$$= -y \sin xy$$

Fundamental trigonometric limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$