Maximum, Minimum and Saddle points

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Def.:- A Function $f(x,y)$ is said to have a Local (relative) maximum value at (x_0, y_0) if there is some region containing (x_0, y_0) in its interior such $f(x,y) \le f(x_0, y_0)$. , $\forall (x,y) \in R$

Def.:- A Function $f(x,y)$ is said to have a Local or (relative) minimum at (x_0, y_0) if there is some region containing (x_0, y_0) in its interior such $f(x,y) \ge f(x_0, y_0)$. , $\forall (x,y) \in R$

Def.:- A point $(x, y, f(x,y))$ in the space $\mathbb{R}^3$ is said to be (سرجية) a Saddle point if it Looks Like a minimum point in Some plane containing it and it Looks Like a maximum point in a nother plane.

Testing for extreme values:-

Def.:- IF $Z = f(x,y)$ is continuous, the extreme value of f may accure only at :-

1. Boundary point at the domain of f.
2. Interior point of the domain of f where $f_x = f_y = 0$

Theorem :- Suppose that $f(x,y)$ is defined in a region R containing (x_0, y_0) in its interior.

Suppose $f_x\big|_{P_0}$, $f_y\big|_{P_0}$ are defined and that $f(x,y) \le f(x_0, y_0)$ For all (x, y) in R that is $f(x_0, y_0)$ is a relative or Local max. then $f_x\big|_{P_0} = f_y\big|_{P_0} = 0$.

Corollary :- The some result hold at a relative min.

Def :- A value (x_0, y_0) at which both f_x, f_y vanish is calle a critical point of f.

Theorem (Second derivative Test).

IF f has continuous First and second order partial derivative on Some open disk containing (a,b) and if $f_x\big|_{(a,b)} = f_y\big|_{(a,b)} = 0$ that is a critical point, then we have.

1. a Local minimum if $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at (a,b)
2. a Local maximum if $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ at (a,b)
3. Saddle point if $f_{xx} f_{yy} - f_{xy}^2 < 0$
4. no-information if $f_{xx} f_{yy} - f_{xy}^2 = 0$

Remark :- The expression $f_{xx}f_{yy}-f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

Ex 1 Find the extreme value of $f(x,y) = 5x^2+4y^2+12$.

sol The domain of f has no-boundary points

$f_x = 10x$, $f_y = 8y$ exists everywhere.

Therefore, relative max. and min can accure only where

$f_x = 10x = 0$, $f_y = 8y = 0$

$\Rightarrow x = 0$, $y = 0$

[⇒ f has extreme value at only point (0,0)]

$f_{xx} = 10$, $f_{yy} = 8$, $f_{xy} = 0$

$\therefore f_{xx}f_{yy} - f_{xy}^2 = (10)(8) - 0 = 80 > 0$ and $f_{xx} = 10 > 0$

⇒ f has a Local minimum

Ex 2 Find the extreme value of $f(x,y) = 8xy$.

sol :- Since the Function is diff. everywhere and its domain has no boundary points, the Function can a ssume extreme vlaue only where $f_x = 8y = 0$, $f_y = 8x = 0$

$\Rightarrow (x,y) = (0,0)$

Therefore f has extreme value at only point (0,0)

$f_{xx} = f_{yy} = 0$, $f_{xy} = 8$.

$f_{xx}f_{yy} - f_{xy}^2 = -64 < 0$ is negetive

$\therefore$ f has a saddle point at (0,0).

Ex: Find the extreme value of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y$

Sol: The Function f is defined and diff. for all x and y and its domain has no boundary points, then the Function can assume extreme value only where $f_x = f_y = 0$.

$\Rightarrow f_x = y - 2x - 2 = 0 \Rightarrow y - 2x - 2 = 0 \quad \dots (1)$

$f_y = x - 2y - 2 = 0 \Rightarrow x - 2y - 2 = 0 \quad \dots (2)$

Let $y = 2x + 2$

sub in eq (2) to get $x - 2(2x+2) - 2 = x - 4x - 4 - 2 = -3x$

$\Rightarrow -3x - 6 = 0 \Rightarrow \boxed{x = -2}$

$\therefore y = -4 + 2 = -2 \rightarrow (-2,-2)$ is extreme point

$f_{xx} = -2 \quad , f_{yy} = -2 \quad , f_{xy} = 1.$

$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 1 = 4 - 1 = 3 > 0$ and $f_{xx} = -2 < 0$

Then f has a local max. at $(-2,-2)$

The value of f at $(-2,-2) = 8$.

Exercises: Find the local maximum, local minimum and saddle point of each of the following function

(1) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

(2) $f(x,y) = x^2 + xy + 3x + 2y + 5$

(3) $f(x,y) = -2x^2 - y^2 - 2xy + 2x + 2y + 3$