Lagrange Multipliers :-

Suppose $f(x,y,z)$ and $g(x,y,z)$ have continuous partial derivatives

To find the Local max., min. value of $f(x,y,z)$ subject to the constraints $g(x,y,z)=0$.

Find the value of x, y, z and λ that satisfy the eq.

$\vec{\nabla f} = \lambda . \vec{\nabla g}$ where the number λ is called Lagrange Multipl

$f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$ — لـ يجاد النقاط الحرجة

$g(x,y,z) = 0$

Ex 1 Find the min. of $f(x,y,z) = 3x + 2y + z + 5$ subject to

$g(x,y,z) = 9x^2 + 4y^2 - z = 0$

Sol

$$\vec{\nabla f} = \lambda . \vec{\nabla g} \quad ---- *$$

$$\vec{\nabla f} = f_x i + f_y j + f_z k = 3i + 2j + k.$$

$$\vec{\nabla g} = g_x i + g_y j + g_z k = 18x i + 8y j - k.$$

Sub in $*$ To get :-

$$3i + 2j + k = \lambda(18x i + 8y j - k).$$

$$3 = \lambda(18x)$$

$$2 = \lambda(8y)$$

$$\lambda = -1$$

and $3 = -18x \Rightarrow x = \frac{-3}{18} = -\frac{1}{6}$

$2 = -8y \Rightarrow y = \frac{2}{-8} = -\frac{1}{4}$

$\therefore g(x,y,z) = 9x^2 + 4y^2 - z = \frac{9}{36} + \frac{4}{16} - z = 0 \Rightarrow z = \frac{9}{36} + \frac{4}{16} = \frac{1}{2}$

$\therefore$ The point $(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2})$ has min point and $f(-\frac{1}{6}, \frac{-1}{4}, \frac{1}{2}) = \frac{9}{2}$

Ex. Use Method of Lagrange Multipliers to find

1. The min. value of $x+y$ subject to the constraints $xy = 16$, $x > 0$, $y > 0$
2. the max value of xy subject to the constraints $x+y = 16$.

Sol ① $f(x,y) = x+y$, $g(x,y) = xy - 16 = 0$

$\vec{\nabla f} = i + j$, $\vec{\nabla g} = yi + xj$

$\therefore \vec{\nabla f} = \lambda \cdot \vec{\nabla g}$

$\Rightarrow i + j = \lambda(yi + xj)$

$\Rightarrow y\lambda = 1 \Rightarrow \lambda = \frac{1}{y}$ --- ①

$\lambda x = 1$ --- ② $\xRightarrow{(1) \text{ in } (2)} \frac{1}{y} x = 1$

$\Rightarrow \frac{x}{y} = 1 \rightarrow \boxed{x = y}$

$\therefore g(x,y) \overset{(x=y)}{=} x^2 - 16 = 0 \rightarrow x = \mp 4 \Rightarrow y = \mp 4$

$\therefore (4,4), (-4,-4)$, $\because x > 0$, $y > 0$

f has min value at $(4,4) \Rightarrow f(4,4) = 8$.

② $f(x,y) = xy$, $g(x,y) = x + y - 16$

$\vec{\nabla f} = yi + xj$, $\vec{\nabla g} = i + j$

$\therefore \vec{\nabla f} = \lambda \vec{\nabla g} \Rightarrow yi + xj = \lambda(i + j)$

$\Rightarrow y = \lambda$ --- ①

$x = \lambda$ --- ②

We get $\boxed{x = y}$

$\therefore g(x,y) = x + x - 16 = 0 \Rightarrow 2x - 16 = 0 \Rightarrow x = 8$ and $y = 8$

$\therefore (8,8)$ is the max. point

$\therefore f$ has max. point at $(8,8)$ and $f(8,8) = 64$.

Implicit Differentiation

الاشتقاق الضمني

Theorem(1) :- suppose that the equation $f(x,y,z)=0$ determines z as a differentiable function of x and y. Then at points where $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example (1) :- Suppose that x, y, and z are variables and z is a function of x and y satisfying that $2x^2+y^2+z^2-25=0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(2,-3,3)$

solution:

$$f(x,y,z) = 2x^2+y^2+z^2-25=0$$

$$F_x = 4x \quad , \quad F_y = 2y \quad , \text{and} \quad F_z = 2z$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x}{2z} = \frac{-2x}{z}$$

and $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-2y}{2z} = -\frac{y}{z}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(2,-3,3)} = -2\left(\frac{2}{3}\right) = -\frac{4}{3}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,-3,3)} = -\left(\frac{-3}{3}\right) = 1$$

Example (2): suppose that x, y and z are variables and z is a function of x and y satisfying that $x^3+y^3+z^3+3xyz=2$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(2,1,-1)$

Solution: H.w.

Example (3): suppose that r, s, t and w are variables and w is a function of r, s, and t satisfying that $e^{rt}-2se^{w}+wt-3w^2r=5$. Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$, and $\frac{\partial w}{\partial t}$

Solution:

$$F(r,s,t,w)=e^{rt}-2se^{w}+wt-3w^2r-5$$

$\Rightarrow F_r=te^{rt}-3w^2$, $F_s=-2e^{w}$, $F_t=re^{rt}+w$

$F_w=-2se^{w}+t-6wr$

$$\Rightarrow \frac{\partial w}{\partial r}=-\frac{F_r}{F_w}=\frac{-te^{rt}+3w^2}{-2se^{w}+t-6wr},$$

$$\frac{\partial w}{\partial s}=-\frac{F_s}{F_w}=\frac{2e^{w}}{-2se^{w}+t-6wr}, \text{ and}$$

$$\frac{\partial w}{\partial t}=-\frac{F_t}{F_w}=\frac{re^{rt}+w}{2se^{w}-t+6wr}$$