Fubini's theorem (First form)

If $f(x,y)$ is continous on the rectangular
$R: a \leq x \leq b\ ,\ c \leq y \leq d$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex Calcalate $\iint_R f(x,y)\,dA$ for $f(x,y) = 1-6x$
and $R: 0 \leq x \leq 2\ ,\ -1 \leq y \leq 1$

Solution

By Fubini's theorem

$$\iint_R f(x,y)\,dA = \int_{y=-1}^{y=1} \int_{x=0}^{x=2} (1-6x^2y)\,dx\,dy = \int_{-1}^{1} x - 2x^3y \Big]_{x=0}^{x=2}$$

$$= \int_{-1}^{1} (2-16y)\,dy = \Big[2y - \tfrac{16}{2}y^2\Big]_{-1}^{1} = \Big[2y - 8y^2\Big]_{-1}^{1}$$

$$=(2-8)-(-2-8) = 2-8+2+8 = 4$$

Reversing the over of integration gives the same ans

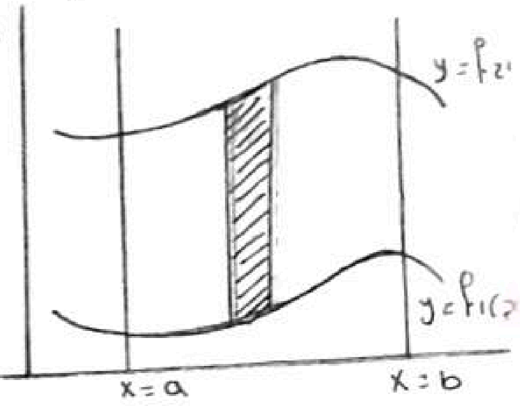
$$\int_0^2 \int_{-1}^{1} (1-6x^2y)\,dy\,dx = \int_0^2 \Big[y-3x^2y^2\Big]_{-1}^{1} dx = \int_0^2 [1-3x^2]-[-1-3x$$

$$= \int_0^2 2\,dx = 2x\Big|_0^2 = 4$$

...inis theorem (stronger Form)

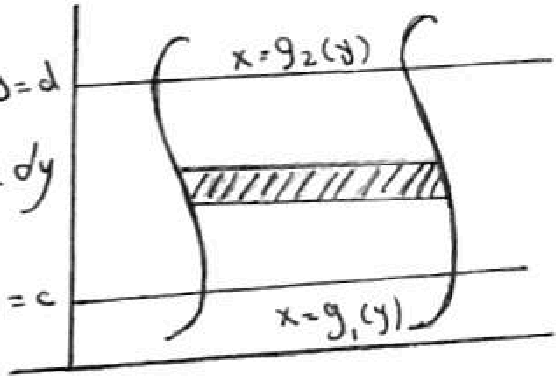
Let $f(x,y)$ be a continous on a region R
if R is defined by $a \leq x \leq b$, $f_1(x) \leq y \leq f_2(x)$ with f_1 and f_2 continous on $[a,b]$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$$



if R is defined by $c \leq y \leq d$, $g_1(y) \leq x \leq g_2(y)$ with g_1, g_2 are continous on $[c,d]$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$



Remark:- if we take $f(x,y) = 1$ in the theorem (Fubini's theorem, stronger Form) then the area of the region R defind by

$a \leq x \leq b$, $f_1(x) \leq y \leq f_2(y)$ with f_1, f_2 are continuous on $[a,b]$ then

$$\text{Area} = \iint_R dA = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

if R defind by $c \leq y \leq d$, $g_1(y) \leq x \leq g_2(y)$ with g_1, g_2 are continous on $[a,b]$ then the area of R

$$\text{Area} = \iint_R dA = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\,dy$$

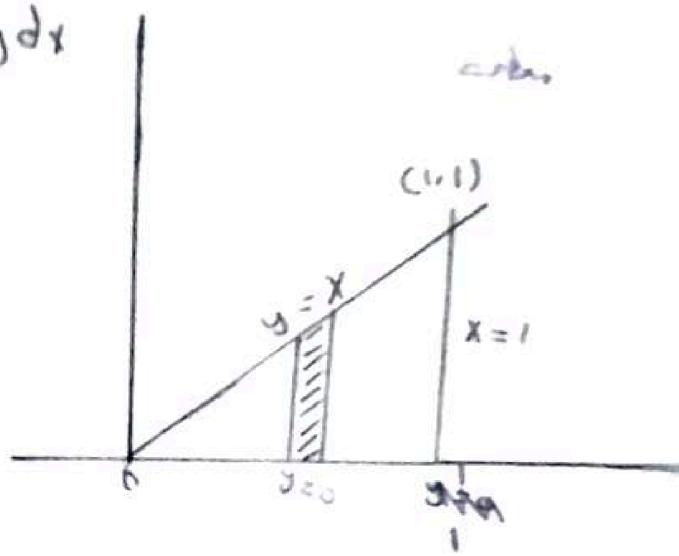
Ex Find the volume of the prism whose base is the triangle in the xy plane bounded by the x-axis and the line y = x and x = 1 and whose top lies in the plane $z = f(x,y) = 3 - x - y$

Solution (1-) $V = \displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=x} (3 - x - y)\, dy\, dx$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \frac{3x^2}{2} - \frac{x^3}{2}\Big]_0^1 = 1$$



(2-) When the over of integration is reversed the integral for the volum is

$$V = \int_{y=0}^{y=1}\int_{x=y}^{x=1} (3 - x - y)\, dx\, dy$$

$$= \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2\right) dy$$

$$= \frac{5}{2} y - 2y^2 + \frac{y^3}{2}\Big]_0^1 = 1$$