Triple Integral

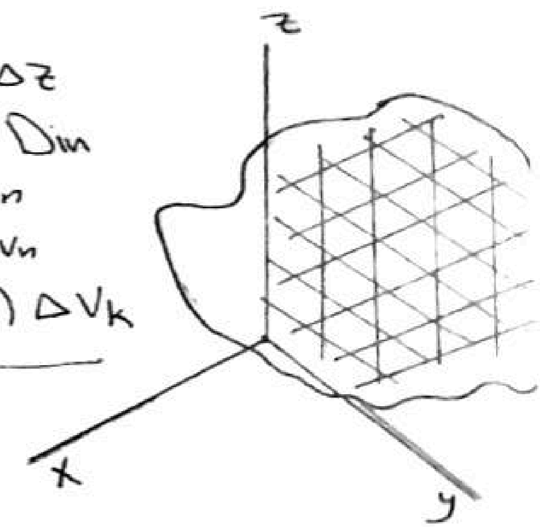
If $f(x,y,z)$ is a function defined on bounded region in a space then the integral of f over D may be defin in the following way we partition orectangular regio about D into rectangular cells by planes parallel to the coordinate planes as shown

the cells have dimensions $\Delta x, \Delta y, \Delta z$

we number the cells that lie inside D in some order $\Delta V_1, \Delta V_2, \Delta V_3, \dots \Delta V_n$

shoose apoint (x_k, y_k, z_k) in each ΔV_n

from the sum $S_n = \sum_{n=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$



If f is cont. then $\Delta x, \Delta y, \Delta z$ all approch zero then the sums S_n will approch a limit

$$\lim S_n = \iiint_D f(x,y,z)\,dv$$

we call this limit the Triple integral of f over D

$$\text{then } \iiint_D f(x,y,z)\,dv = \int_{x=a_1}^{x=a_2} \int_{y=f_1(x)}^{y=f_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

or

$$\iiint_D f(x,y,z)\,dv = \iint_R \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz \right) dy\,dx$$

Proporties of Triple integral

1- $\iiint_D K f \, dv = K \iiint_D f \, dv$ (K is any number

2. $\iiint_D [f \mp g] \, dv = \iiint_D f \, dv \mp \iiint_D g \, dv$

3- $\iiint_D f \, dv \geqslant 0$ if $f \geqslant 0$ on D

4- $\iiint_D f \, dv \geqslant \iiint_D g \, dv$ if $f \geqslant g$ on D

5- $\iiint_D f \, dv = \iiint_{D_1} f \, dv + \iiint_{D_2} f \, dv + \iiint_{D_3} f \, dv + \cdots + \iiint_{D_n} f \, dv$

Volume

If $f(x, y, z) = 1$ is the constant function Whose value is one then the sums reduce to

$$S_n = \sum_{k=1}^{n} 1 \cdot \Delta V_k = \sum_{k=1}^{n} \Delta V_k$$

$\therefore$ volume of $D = \lim \sum_{k=1}^{n} \Delta V_k$

$$= \iiint_D dv = \iint_R \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} dz \right) dx \, dy$$

and the volume of a closed bounded region D in space is the value of the integral

$$\text{volume of } D = \iiint_D dv$$

Ex ① Evaluate $\displaystyle\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\, dz\, dy\, dx$

Solution $\displaystyle\int_{x=-2}^{x=5}\int_{y=0}^{y=3x} 4z\Big]_{z=y}^{z=x+2} dy\, dx = \int_{-2}^{5}\int_{0}^{3x} (4(x+2)-4y)\, dy\, dx$

$$= \int_{-2}^{5}\left[4(x+2)y - \frac{4y^2}{2}\right]_0^{3x} dx = \int_{-2}^{5} 4(x+2)(3x)\,dx - 2\int_{-2}^{5} 9x^2\,dx =.$$

$$= \int_{-2}^{5} 12x^2 + 24x - 18x^2\, dx$$

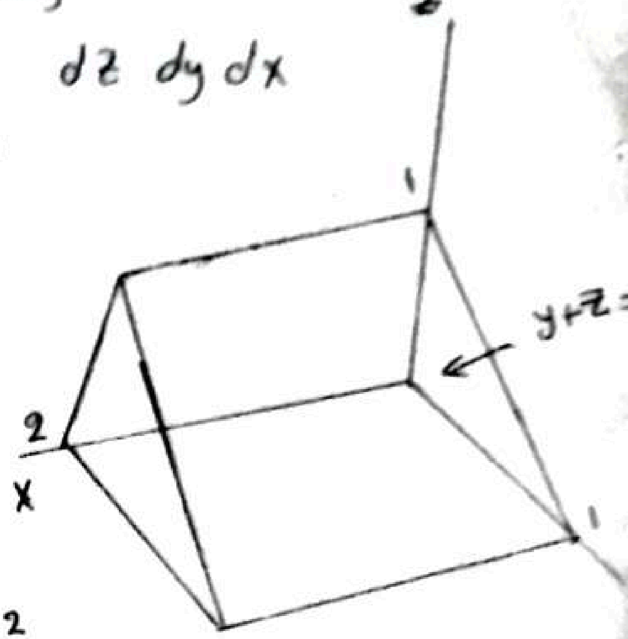
Ex ② Evaluate $\displaystyle\int_{z=0}^{z=1}\int_{y=0}^{y=1-z}\int_{x=0}^{x=2} dx\, dy\, dz$

Solution $\displaystyle\int_0^1\int_0^{1-z} x\Big]_0^2 dy\, dz = \int_0^1\int_0^{1-z} 2\, dy\, dz$

$$= \int_0^1 [2y]_0^{1-z}\, dz = \int_0^1 2(1-z)\,dz = [2z - z^2]_0^1 = 2-1 = 1$$

Ex ③ Find the volume of the solid shown bellow

Solution $V = \int_{0}^{2} \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} dz\, dy\, dx$



$= \int_0^2 \int_0^1 z \Big]_0^{1-y} dy\, dx$

$= \int_0^2 \int_0^1 (1-y)\, dy\, dx = \int_0^2 \int_0^1 (1-y)\, dy\, dx$

$= \int_0^2 (y - \frac{y^2}{2}) \Big]_0^1 dx = \int_0^2 (1 - \frac{1}{2}) dx = \int_0^2 \frac{1}{2} dx = \frac{1}{2} x \Big]_0^2 = \underline{\underline{1}}$

Ex ④ Find the volume of rectangular Solid in the First octant bounded by the coordinate planes and the planes $x=1, y=2$ and $z=3$.

Solution $\int_0^2 \int_0^1 \int_0^3 dz\, dx\, dy = \int_0^2 \int_0^1 z \Big]_0^3 dx\, dy = \int_0^2 \int_0^1 3\, dx\, d$

$= \int_0^2 3x \Big]_0^1 dy = \int_0^2 3\, dy = 3y \Big]_0^2 = \underline{\underline{6}}$

also $\int_0^3 \int_0^1 \int_0^2 dy\, dx\, dz = \int_0^3 \int_0^1 y \Big]_0^2 dx\, dz = \int_0^3 \int_0^1 2\, dx\, dz$

$= \int_0^3 2x \Big]^1 dz = \int_0^3 2\, dz = 2z \Big]_0^3 = \underline{\underline{6}}$