Remark

$$dy\,dx\,dz = dx\,dy\,dz = dx\,dz\,dy = dy\,dz\,dx$$
$$= dz\,dx\,dy = dz\,dy\,dx.$$

Ex ⑤ Find the volume in the First octant enclosed by the cylinder $x^2+z^2=4$ and the plane $y=3$.

Solution

$x = \mp 2$

$$V = \int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy = \int_0^3\int_0^2 z\Big]_0^{\sqrt{4-x^2}}$$

$$= \int_0^3\int_0^2 \sqrt{4-x^2}\,dx\,dy \Rightarrow \quad \text{Let } \boxed{x = 2\sin\theta} \Rightarrow \sin\theta = \frac{x}{2} \Rightarrow$$

$$\theta = \sin^{-1}\frac{x}{2}, \quad \boxed{dx = 2\cos\theta\,d\theta}$$

$$4-x^2 = 4-4\sin^2\theta \Rightarrow 4(1-\sin^2\theta) = \boxed{4\cos^2\theta}$$

$$\therefore \int_0^3\int_0^{\frac{\pi}{2}} (2\cos\theta)(2\cos\theta)\,d\theta\,dy = 4\int_0^3\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta\,dy$$

$$= \int_0^3\int_0^{\frac{\pi}{2}} \frac{(1+\cos 2\theta)}{2}\,d\theta\,.dy = 4\int_0^3 \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\Big]_0^{\frac{\pi}{2}} dy$$

$$= 4\int_0^3 \left[\frac{\pi}{4}\right] dy \Rightarrow \cancel{4}\cdot\frac{\pi}{\cancel{4}}\,y\Big]_0^3 = 3\pi$$

Ex ⑥ Find the volume enclosed bytween the two surfaces $z_1 = x^2+3y^2$ and $z_2 = 8-x^2-y^2$.

Solution $x^2+3y^2=8-x^2-y^2$

$2x^2+4y^2=8 \Rightarrow x^2+2y^2=4 \Rightarrow y=\mp\sqrt{\frac{4-x^2}{2}}$

$\therefore x=\mp 2$

$$V=\int_{x=-2}^{x=2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}}\int_{z=x^2+3y}^{z=8-x^2-y^2} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} z\Big]_{x^2+3y^2}^{8-x^2-y^2} dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-2x^2-4y^2)\,dy\,dx$$

$$=\int_{-2}^{2} 8y-2x^2y-\frac{4}{3}y^3\Big]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx=\int_{-2}^{2}(8-2x^2)y-\frac{4}{3}y^3\Big]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$=\int_{x=-2}^{x=2}\left[(8-x^2)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}-\left(-(8-2x^2)\sqrt{\frac{4-x^2}{2}}+\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\right.$$

$$=\int_{-2}^{2} 2(8-2x^2)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left[\frac{4-x^2}{2}\right]^{3/2} dx$$

$$\int_{-2}^{2} 4(4-x^2)\frac{\sqrt{4-x^2}}{\sqrt{2}} - \frac{8}{3}\left[\frac{4-x^2}{2}\right]^{3/2} dx$$

$$= \int_{-2}^{2} 4\,\frac{(4-x^2)^{3/2}}{\sqrt{2}} - \frac{8}{3}\,\frac{(4-x^2)^{3/2}}{2\sqrt{2}}\, dx = \int_{-2}^{2} (4-x^2)^{3/2}\left[\frac{4}{\sqrt{2}} - \frac{4}{3\sqrt{2}}\right] dx$$

$$= \frac{4\sqrt{2}}{3}\int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

let $x = 2\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{2}$

$dx = 2\cos\theta\, d\theta$

$4 - x^2 = 4 - 4\sin^2\theta \Rightarrow 4\cos^2\theta$

$$\therefore \int_{x=-2}^{x=2} (4-x^2)^{3/2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [4\cos^2\theta]^{3/2}\cos\theta . d\theta = 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\, \cos\theta\, d\theta$$

$$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\cos 2\theta)^2}{4}\, d\theta = 2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta = 2\left[\theta + \sin 2\theta + \frac{1}{2}\theta + \frac{1}{8}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} + \frac{\pi}{2} - \left(-\frac{\pi}{2} - \frac{\pi}{4}\right)\right] = 2\left[\pi + \frac{\pi}{2}\right] = 2\left[\frac{3\pi}{2}\right] = 3\pi$$

$$\therefore \frac{4}{3}\sqrt{2}\int_{x=-2}^{x=2} \left[\sqrt{4-x^2}\right]^3 dx = \frac{4}{3}\sqrt{2}\,(3\pi) = 4\sqrt{2}\,\pi.$$