

Chapter Three: Interpolation of Functions

I Lagrange Interpolation

Let $f(x)$ be a real continuous function on $[a, b]$, and we know its value at $x_0, x_1, x_2, \dots, x_n$. Then we can estimate values of $f(x)$,

x	x_0	x_1	x_2	\dots	x_n
$y = f(x)$	y_0	y_1	y_2	\dots	y_n

$$f(x^*) = \sum_{j=0}^n f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^n \frac{(x^* - x_i)}{(x_j - x_i)} ; n=0, 1, 2, \dots$$

x^* = The element ~~which~~ which to find image to it.

Examples:

① By Lagrange formula, find the value of $f(3)$ and $f(5)$, from the table?

x	x_0	x_1	x_2	x_3
	0	1	2	4
$f(x)$	1	1	2	5

Solution: a - $f(3)$, $\therefore x^* = 3$

$$f(x^*) = \sum_{j=0}^n f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^n \frac{(x^* - x_i)}{(x_j - x_i)}$$

$$f(3) = \sum_{j=0}^3 f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^3 \frac{(3 - x_i)}{(x_j - x_i)}$$

$$f(3) = f(x_0) \frac{(3-x_1)(3-x_2)(3-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1) \frac{(3-x_0)(3-x_2)(3-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ + f(x_2) \frac{(3-x_0)(3-x_1)(3-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3) \frac{(3-x_0)(3-x_1)(3-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(3) = (1) \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} + (1) \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} \\ + (2) \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} + (5) \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}$$

$$f(3) = \frac{1}{4} - 1 + 3 + \frac{5}{4}$$

$$f(3) = 3.5$$

b. $f(5) \rightarrow x^* = 5$

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$$f(x^*) = f(5) = f(x_0) \frac{(5-x_1)(5-x_2)(5-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1) \frac{(5-x_0)(5-x_2)(5-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ + f(x_2) \frac{(5-x_0)(5-x_1)(5-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3) \frac{(5-x_0)(5-x_1)(5-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(5) = (1) \frac{(5-1)(5-2)(5-4)}{(0-1)(0-2)(0-4)} + (1) \frac{(5-0)(5-2)(5-4)}{(1-0)(1-2)(1-4)}$$

$$+ (2) \frac{(5-0)(5-1)(5-4)}{(2-0)(2-1)(2-4)} + (5) \frac{(5-0)(5-1)(5-2)}{(4-0)(4-1)(4-2)}$$

$$f(5) =$$

$$f(5) = 6$$

ex: find the value of $f(2.1)$, where $x_0 = 1.1$, $x_1 = 1.7$
 $x_2 = 3$ and $y_0 = 10.6$, $y_1 = 15.2$, $y_2 = 20.3$?

$$x^* = 2.1$$

$$f(x^*) = f(2.1) = \sum_{j=0}^n f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^n \frac{(x^* - x_i)}{(x_j - x_i)}$$

$$f(2.1) = f(x_0) \frac{(2.1 - x_1)(2.1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(2.1 - x_0)(2.1 - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

~~$$+ f(x_2) \frac{(2.1 - x_0)(2.1 - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$~~

$$+ f(x_2) \frac{(2.1 - x_0)(2.1 - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$f(2.1) = (10.6) \frac{(2.1 - 1.7)(2.1 - 3)}{(1.1 - 1.7)(1.1 - 3)} + (15.2) \frac{(2.1 - 1.1)(2.1 - 3)}{(1.7 - 1.1)(1.7 - 3)}$$

$$+ (20.3) \frac{(2.1 - 1.1)(2.1 - 1.7)}{(3 - 1.1)(3 - 1.7)}$$

$$f(2.1) =$$

Exc. from the table

x	1	4	5
$f(x)$	16	88	0

Find (1) $f(0)$ (2) $f(2)$?

II Inverse of Lagrange interpolation

$$x^* = \sum_{j=0}^n x_j \frac{\prod_{i=0, i \neq j}^n (y^* - y_i)}{(y_j - y_i)}$$

exe: find the value of x^* , when $y^* = 2$ from the table

x	15	20	2	?
$y = f(x)$	1	3	5	

Solution.

$$x^* = \sum_{j=0}^n x_j \frac{\prod_{i=0, i \neq j}^n (y^* - y_i)}{(y_j - y_i)}$$

$$x^* = x_0 \frac{(2 - y_1)(2 - y_2)}{(y_0 - y_1)(y_0 - y_2)} + x_1 \frac{(2 - y_0)(2 - y_2)}{(y_1 - y_0)(y_1 - y_2)} + x_2 \frac{(2 - y_0)(2 - y_1)}{(y_2 - y_0)(y_2 - y_1)}$$

$$x^* = (15) \frac{(2-3)(2-5)}{(1-3)(1-5)} + (20) \frac{(2-1)(2-5)}{(3-1)(3-5)} + (2) \frac{(2-1)(2-3)}{(5-1)(5-3)}$$

$$x^* = \frac{45}{8} + 15 - \frac{1}{4}$$

$$= \frac{163}{8}$$

$$x^* = 20.375$$

ex. Use the following table to find x^*
 where $y^* = 0.2703$

y	0.625	0.3443	0.2025
x	1.6	2.9	4.3

$$x^* = \frac{\sum_{j=0}^2 x_j}{\sum_{\substack{i=0 \\ i \neq j}}^2 (y_j - y_i)}$$

$$= x_0 \frac{(0.2703 - y_1)(0.2703 - y_2)}{(y_0 - y_1)(y_0 - y_2)} + x_1 \frac{(0.2703 - y_0)(0.2703 - y_2)}{(y_1 - y_0)(y_1 - y_2)} + x_2 \frac{(0.2703 - y_0)(0.2703 - y_1)}{(y_2 - y_0)(y_2 - y_1)}$$

$$x^* = (1.6) \frac{(0.2703 - 0.3443)(0.2703 - 0.2025)}{(0.625 - 0.3443)(0.625 - 0.2025)} + (2.9) \frac{(0.2703 - 0.625)(0.2703 - 0.2025)}{(0.3443 - 0.625)(0.3443 - 0.2025)}$$

$$+ (4.3) \frac{(0.2703 - 0.625)(0.2703 - 0.3443)}{(0.2025 - 0.625)(0.2025 - 0.3443)}$$

=

$$x^* = 3.8658$$