CH5: Numerical Differentiation

When we don't have an explicit function f of x, but we have only a given data of n+1 points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ where $y_i = f(x_i)$, and in the same time we want to find the values of $f'(x)$ and $f''(x)$ at some values of x in the interval $[x_0, x_n]$ without knowing the exact formula of f, we should use the following formulas depending on the position of x in the interval $[x_0, x_n]$, where the forward difference formulas will be used for the values of x in the first half of the interval $[x_0, x_n]$ while the backward difference formulas will be used for the values of x in the last half of the interval of $[x_0, x_n]$:

عندما لا يكون لدينا صيغة صريحة للدالة f للمتغير x وإنما يتوفر لدينا $n+1$ من النقاط المعلومة الواقعة على المنحنى $y=f(x)$ وهي (x_0,y_0) و (x_1,y_1) و ..... و (x_n,y_n) حيث أن $y_i=f(x_i)$ وفي نفس الوقت نرغب بإيجاد قيم المشتقات $f'(x)$ و $f''(x)$ عند قيم معينة لـ x في الفترة $[x_0,x_n]$ فعلينا حينها استخدام الصيغ the formulas التالية إعتماداً على موقع x في الفترة $[x_0,x_n]$ حيث إننا

سنستخدم ال forward difference formulas لايجاد $f'(x)$ و $f''(x)$ لقيم x الواقعة في النصف الأول من الفترة $[x_0, x_n]$ بينما سنستخدم ال backward difference formulas لإيجاد المشتقات $f'(x)$ و $f''(x)$ لقيم x الواقعة في النصف الأخير من الفترة $[x_0, x_n]$.

1) Forward Difference Formulas :

$$f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + (2k-1)\frac{\Delta^2 y_0}{2!} + (3k^2-6k+2)\frac{\Delta^3 y_0}{3!} + (4k^3-18k^2+22k-6)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

where $h = x_{i+1} - x_i$, $k = \dfrac{x - x_0}{h}$,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (k-1)\Delta^3 y_0 + \left(\frac{1}{2}k^2 - \frac{3}{2}k + \frac{11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

2) Backward Difference Formulas:

$$f'(x) = \frac{dy}{dx} = \frac{1}{h}\Big[\nabla y_n + (2k+1)\frac{\nabla^2 y_n}{2!}$$

$$+ (3k^2+6k+2)\frac{\nabla^3 y_n}{3!}$$

$$+ (4k^3+18k^2+22k+6)\frac{\nabla^4 y_n}{4!} + \cdots\Big]$$

where $h = x_{i+1} - x_i$, $k = \frac{x - x_n}{h}$,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\Big[\nabla^2 y_n + (k+1)\nabla^3 y_n$$

$$+ \left(\frac{1}{2}k^2 + \frac{3}{2}k + \frac{11}{12}\right)\nabla^4 y_n + \cdots\Big]$$

Example 1: Find the derivatives

$f'(0.2)$, $f''(0.2)$, $f'(3.7)$, $f''(3.7)$ by

using the following data

x_i	0	1	2	3	4
y_i	0	0	8	54	192

Solution:

1) For $x = 0.2$

$h = x_{i+1} - x_i = 1$, $x = 0.2$, $x_0 = 0$,

$$k = \frac{x - x_0}{h} = \frac{0.2 - 0}{1} = 0.2 .$$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	0				
		0			
1	0		8		
		8		30	
2	8		38		24
		46		54	
3	54		92		
		138			
4	192				

$$f'(0.2) = \frac{1}{h}\Big[\Delta y_0 + (2k-1)\frac{\Delta^2 y_0}{2!} + (3k^2 - 6k + 2)\frac{\Delta^3 y_0}{3!}$$

$$+ (4k^3 - 18k^2 + 22k - 6)\frac{\Delta^4 y_0}{4!}\Big]$$

$$= \frac{1}{1}\Big[0 + (2*(0.2) - 1) * \frac{8}{2}$$

$$+ (3*(0.2)^2 - 6*(0.2) + 2) * \frac{30}{6}$$

$$+ (4*(0.2)^3 - 18*(0.2)^2 + 22*(0.2) - 6) * \frac{24}{24}\Big]$$

$$= 0 - 2.4 + 4.6 - 2.288 = -0.088$$

$$f''(0.2) = \frac{1}{h^2}\left[\Delta^2 y_0 + (k-1)\Delta^3 y_0 + \left(\frac{1}{2}k^2 - \frac{3}{2}k + \frac{11}{12}\right)\Delta^4 y_0\right]$$

$$= \frac{1}{1}\left[8 + (0.2-1)*30 + \left(\frac{1}{2}*(0.2)^2 - \frac{3}{2}*(0.2) + \frac{11}{12}\right)*24\right]$$

$$= 8 - 24 + 15.28 = -0.72$$

2) For $x = 3.7$

$h = x_{i+1} - x_i = 1$, $x = 3.7$, $x_4 = 4$

$$k = \frac{x - x_4}{h} = \frac{3.7-4}{1} = -0.3$$

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
0	0				
		0			
1	0		8		
		8		30	
2	8		38		24
		46		54	
3	54		92		
		138			
4	192				

$$f'(3.7) = \frac{1}{h}\left[\nabla y_4 + (2k+1)\frac{\nabla^2 y_4}{2!} + (3k^2+6k+2)\frac{\nabla^3 y_4}{3!} + (4k^3+18k^2+22k+6)\frac{\nabla^4 y_4}{4!}\right]$$

$$= \frac{1}{1}\left[138 + (2*(-0.3)+1)*\frac{92}{2}\right.$$

$$+ (3*(-0.3)^2 + 6*(-0.3)+2)*\frac{54}{6}$$

$$\left.+ (4*(-0.3)^3 + 18*(-0.3)^2 + 22*(-0.3)+6)*\frac{24}{24}\right]$$

$$= 138 + 18.4 + 4.23 + 0.912 = 161.542$$

$$f''(3.7) = \frac{1}{h^2}\left[\nabla^2 y_4 + (k+1)\nabla^3 y_4\right.$$

$$\left.+ \left(\frac{1}{2}k^2 + \frac{3}{2}k + \frac{11}{12}\right)\nabla^4 y_4\right]$$

$$= \frac{1}{1}\left[92 + (-0.3+1)*54\right.$$

$$\left.+ \left(\frac{1}{2}*(-0.3)^2 + \frac{3}{2}*(-0.3) + \frac{11}{12}\right)*24\right]$$

$$= 92 + 37.8 + 12.28 = 142.08$$

Example 2: Find the derivatives $f'(3.2)$, $f''(3.2)$, $f'(7.6)$, $f''(7.6)$ by using the following data

x_i	1	3	5	7	9
y_i	1	7	21	43	73

Solution:

1) For $x = 3.2$

$h = x_{i+1} - x_i = 2$, $x = 3.2$, $x_0 = 1$,

$$k = \frac{x - x_0}{h} = \frac{3.2 - 1}{2} = \frac{2.2}{2} = 1.1$$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
1	1				
		6			
3	7		8		
		14		0	
5	21		8		0
		22		0	
7	43		8		
		30			
9	73				

$$f'(3.2) = \frac{1}{h}\left[\Delta y_0 + (2k-1)\frac{\Delta^2 y_0}{2!} + (3k^2 - 6k + 2)\frac{\Delta^3 y_0}{3!} + (4k^3 - 18k^2 + 22k - 6)\frac{\Delta^4 y_0}{4!}\right]$$

$$= \frac{1}{2}\left[6 + (2*(1.1) - 1) * \frac{8}{2} + (3*(1.1)^2 - 6*(1.1) + 2) * \frac{0}{6} + (4*(1.1)^3 - 18*(1.1)^2 + 22*(1.1) - 6) * \frac{0}{24}\right]$$

$$= \frac{1}{2}\left[6 + 4.8 + 0 + 0\right] = \frac{1}{2} * 10.8 = 5.4$$

$$f''(3.2) = \frac{1}{h^2}\Big[\Delta^2 y_0 + (k-1)\,\Delta^3 y_0 + \left(\frac{1}{2}k^2 - \frac{3}{2}k + \frac{11}{12}\right)\Delta^4 y_0\Big]$$

$$= \frac{1}{4}\Big[8 + (1.1-1) * 0 + \left(\frac{1}{2} * (1.1)^2 - \frac{3}{2} * (1.1) + \frac{11}{12}\right) * 0\Big]$$

$$= \frac{1}{4} * 8 = 2$$

2) For $x = 7.6$

$h = x_{i+1} - x_i = 2$, $x = 7.6$, $x_4 = 9$,

$$k = \frac{x - x_4}{h} = \frac{7.6 - 9}{2} = \frac{-1.4}{2} = -0.7$$

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
1	1				
		6			
3	7		8		
		14		0	
5	21		8		0
		22		0	
7	43		8		
		30			
9	73				

$$f'(7.6) = \frac{1}{h}\Big[\nabla y_4 + (2k+1)\frac{\nabla^2 y_4}{2!} + (3k^2+6k+2)\frac{\nabla^3 y_4}{3!} + (4k^3+18k^2+22k+6)\frac{\nabla^4 y_4}{4!}\Big]$$

$$= \frac{1}{2}\Big[30 + (2*(-0.7)+1)*\frac{8}{2}$$

$$+ (3*(-0.7)^2 + 6*(-0.7)+2)*\frac{0}{3!}$$

$$+ (4*(-0.7)^3 + 18*(-0.7)^2 + 22*(-0.7)+6)*\frac{0}{4!}\Big]$$

$$= \frac{1}{2}[30 - 1.6 + 0 + 0] = \frac{1}{2}*28.4 = 14.2$$

$$f''(7.6) = \frac{1}{h^2}\Big[\nabla^2 y_4 + (k+1)\nabla^3 y_4$$

$$+ \left(\frac{1}{2}k^2 + \frac{3}{2}k + \frac{11}{12}\right)\nabla^4 y_4\Big]$$

$$= \frac{1}{4}\Big[8 + (-0.7+1)*0 + \left(\frac{1}{2}*(-0.7)^2 + \frac{3}{2}*(-0.7) + \frac{11}{12}\right)*0\Big]$$

$$= \frac{1}{4}[8+0+0] = \frac{1}{4}*8 = 2.$$

Exercise: Find the derivatives

$f'(2.2)$, $f''(2.2)$, $f'(9.3)$, $f''(9.3)$

by using the following data

x_i	2	4	6	8	10
y_i	2	1	3	8	20