S2: Simpson's Rule

This rule is considered as more accurate than the trapezoidal rule to calculate the definite integral $I=\int_a^b f(x)\,dx$ and this rule will be used in the following two cases

هذه القاعدة (قاعدة سمبسون) تعتبر أدق من قاعدة شبه المنحرف لاحتساب التكامل المحدود $I=\int_a^b f(x)\,dx$ وهذه القاعدة سوف تستخدم في الحالتين التاليتين :

1) When the formulla of $f(x)$ is not known and we have only $n+1$ known points on the curve $y=f(x)$ which are $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ satisfying that $x_{i+1}-x_i=h$, $i=0,1,\dots,n-1$ where h is a fixed number and n is an even number, then

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[y_0+y_n+2(y_2+y_4+\dots+y_{n-2})+4(y_1+y_3+\dots+y_{n-1})\Big].$$

عندما تكون الصيغة للدالة $f(x)$ غير معروفة ولكن لدينا $n+1$ من النقاط الواقعة على المنحني $y=f(x)$ وهي النقاط (x_0, y_0) و (x_1, y_1) و ... و (x_n, y_n) بحيث أن $x_{i+1}-x_i=h$ لكل $i=0,1,\dots,n-1$ وأن h عدد ثابت وأن n عدد زوجي . لذا يكون

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2})$$
$$+ 4(y_1 + y_3 + \cdots + y_{n-1})\Big]$$

Example: By using the Simpson's rule, find the value of $\int_1^3 f(x)\,dx$,

where the graph of $y = f(x)$ pass through the points $(1, 4.2)$, $(1.5, 5.95)$, $(2, 8.2)$, $(2.5, 10.95)$, $(3, 14.2)$.

Solution:

We have the following data table:

	x_i	$y_i = f(x_i)$	
x_0	1	4.2	y_0
x_1	1.5	5.95	y_1
x_2	2	8.2	y_2
x_3	2.5	10.95	y_3
x_4	3	14.2	y_4

$n =$ عدد النقاط $- 1$
$= 5 - 1$
$= 4$

and we have $h = x_{i+1} - x_i = 0.5$. Then

$$I = \int_1^3 f(x)\,dx \approx \frac{h}{3}\left[y_0 + y_4 + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{0.5}{3}\left[4.2 + 14.2 + 2(8.2) + 4(5.95 + 10.95)\right]$$

$$= \frac{1}{6} * 102.4 = 17.06667.$$

2) When the formula of $f(x)$ is known and defined on the interval $[a, b]$ and we want to evaluate an approximate value of $I = \int_a^b f(x)\,dx$ by using the Simpson's rule, then we will follow the following steps:

1. Divide the interval $[a, b]$ into n equal parts, each one of these parts of length $h = \frac{b-a}{n}$, where n is an even number.

2. Find the values of $y_i = f(x_i)$, for each $i = 0, 1, \dots, n$, where $x_i = a + i * h$

3. Construct the table of data that you get as follows:

x_i	$y_i = f(x_i)$
$a = x_0$	y_0
x_1	y_1
$\vdots$	$\vdots$
$b = x_n$	y_n

4. Use the Simpson's formula to find the value of the definite integral

$$I = \int_a^b f(x)\,dx .$$

Example: By using the Simpson's rule, find the approximate value of

$$I = \int_0^2 x^2 e^{-x^2}\,dx \quad , \text{when } n = 8 .$$

Solution:

Divide the interval to eight equal parts to get $h = \frac{2-0}{8} = \frac{1}{4} = 0.25$

	x_i	$y_i = f(x_i) = x_i^2 e^{-x_i^2}$	
x_0	0	0	y_0
x_1	0.25	0.058713316	y_1
x_2	0.5	0.194700195	y_2
x_3	0.75	0.320502838	y_3
x_4	1	0.367879441	y_4
x_5	1.25	0.327517792	y_5
x_6	1.5	0.237148255	y_6
x_7	1.75	0.143235031	y_7
x_8	2	0.073262555	y_8

$$I = \frac{h}{3}\left[y_0 + y_8 + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)\right]$$

$$= \frac{0.25}{3}\left[0+0.073262555+2(0.194700195\right.$$

$$+0.367879441+0.237148255)$$

$$+4(0.058713316+0.320502838$$

$$\left.+0.327517792+0.143235031)\right]$$

$$=0.08333333 * 5.072594245$$

$$=0.422716187$$

Exercise:

1) By using the Simpson's rule, find the value of $\int_0^2 f(x)\,dx$, where the graph of $y=f(x)$ pass through the points $(0,1)$, $(0.5, 2.149)$, $(1, 3.718)$, $(1.5, 5.982)$, $(2, 9.389)$.

2) By using the Simpson's rule, find the approximate value of

$$I = \int_{-2}^{2} x^2 e^x \, dx, \text{ when } n=6.$$