"Chapter three"

"Estimation"

1- Unbiasedness :-

let $X_1, X_2, \dots, X_n$ be a random sample from a population having pdf $f(x;\theta)$ with unknown parameter θ. let $\hat{\theta}$ be an estimator of θ,

Therefore $\hat{\theta}$ is said to be unbiased estimator of θ, iff $E(\hat{\theta}) = \theta$, otherwise, the bias of an estimator $\hat{\theta}$ can be defined as

$$\text{bias}(\theta) = E(\hat{\theta}) - \theta$$

* Properties *

1) $E(k) = k$, k is a constant.
2) $E(kx) = k\,E(x)$
3) $E(kx \mp cy) = kE(x) \mp cE(y)$
   where k and c are constants.
4) $v(k) = 0$
5) $v(kx) = k^2 v(x)$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

* Properties of a Good Estimator :-

An estimator $\hat{\theta}$ is said unbiased if its expected value equals the value of parameter θ being estimated, That is

1) if $E(\hat{\theta}) = \theta$, $\hat{\theta}$ is unbiased
2) if $E(\hat{\theta}) \neq \theta$, $\hat{\theta}$ is biased

" Estimation "

Ex) let $X_i \sim Exp(\theta)$, $i=1,\dots,n$, is $\hat{\theta}=\bar{x}$ is an Unbiased for θ?

Solution)

$$E(\hat{\theta}) = E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

since $X_i \sim Exp(\theta)$ $\therefore E(x_i)=\theta$

$$\rightarrow E(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{n\theta}{n} = \theta$$

$\rightarrow \therefore E(\hat{\theta})=\theta$, Then $\hat{\theta}$ is Unbiased for Parameter θ.

Ex) let $X \sim Gam(\theta,\beta)$ is $\hat{\theta}=\bar{x}$ unbiased for θ?

Solution)

$$E(\hat{\theta}) = E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i)\right)$$

$\because X \sim Gam(\theta,\beta)$ $\therefore E(X)=\theta\beta$

$$\rightarrow E(\hat{\theta}) = \left[\frac{1}{n}\cdot\sum_{i=1}^{n}\theta\beta\right] = \frac{n\theta\beta}{n} = \theta\beta$$

$$\therefore E(\hat{\theta}) \neq \theta$$

So $\hat{\theta}=\bar{x}$ is biased estimator for θ.

Note) 3- The unbiased estimator for above Ex. will be

$$\hat{\theta} = \frac{\bar{x}}{\beta}$$

H.W) let $X_i \sim U(\theta,\theta+1)$, $i=1,2,\dots,n$

Show that $\hat{\theta} = \bar{x}-\frac{1}{2}$ is unbiased estimator for θ?

* Unbias *

Mathematical Expectation:-
let X be a r.v. with Probability function $f(x)$ and $u(x)$ be a real-valued function of X. Then the expected value of $u(x)$ is given by:-

$$E[u(x)] = \begin{cases} \sum_{x=-\infty}^{\infty} u(x) \cdot f(x) & \text{, discrete.} \\ \int_{x=-\infty}^{\infty} u(x) \cdot f(x) & \text{, Continuous} \end{cases}$$

Notes:-

1- if c is a constant, then $E(c) = c$

2- if c is a constant and u is a function of X then $E[c\,u(x)] = c\,E[u(x)]$

3- if $c_1, \dots, c_n$ are constants and $u_1, u_2, \dots, u_n$ are functions, then $E\left(\sum_{i=1}^{n} c_i u_i\right) = \sum_{i=1}^{n} E(c_i u_i)$

$$= \sum_{i=1}^{n} c_i\, E(u_i)$$

Ex) let $X_i \sim U(0,\theta)$, $i = 1,2,3$, is $\hat{\theta} = Y_3$ an unbiased for θ? where Y_3 is O.S. Find the unbiased estimator for θ?

Solution) $E(\hat{\theta}) = E(Y_3)$ $\rightarrow$ Y_3 is O.S.
must find $g(y_3) = ?$

$\because X \sim U(0,\theta) \rightarrow f(x) = \frac{1}{\theta}$, $0 < x < \theta$

Ex) $n=3$ $\quad y_3$ is o.s.

$$f(x)=\frac{1}{\theta} \rightarrow f_x(y_3)=\frac{1}{\theta}$$

$$F(x)=\frac{1}{\theta}\int_0^x f(u)\,du=\frac{x}{\theta} \rightarrow F_x(y_3)=\frac{y_3}{\theta}$$

Then $g(y_3)=3\left[\frac{y_3}{\theta}\right]^2\cdot\frac{1}{\theta}=\begin{cases}\frac{3y_3^2}{\theta^3}, & 0<y_3<\theta\\ 0, & \text{o.w.}\end{cases}$

$$\rightarrow E(y_3)=\int_0^\theta y_3\cdot g(y_3)\,dy_3=\int_0^\theta y_3\cdot\frac{3y_3^2}{\theta^3}\,dy_3$$

$$\rightarrow E(y_3)=\frac{3}{4}\theta \quad \therefore E(\hat{\theta})\neq\theta$$

$\therefore \hat{\theta}=y_3$ is ~~un~~biased estimator for θ.

Note :- The unbiased estimator will be $\hat{\theta}=\frac{4}{3}y_3$.

H.W) let $X_1, X_2, \dots, X_n$ be a r.s. with $E[x_i]=\mu$ and $\sigma(x_i)=\sigma^2$, Show that

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \text{ is unbiased for } \sigma^2?$$

Note :- $E(\bar{x})=\mu$, $\bar{x}$ is unbiased for μ.

$E(S^2)=\sigma^2$, S^2 is unbiased for σ^2.

2) Mean Square Error (MSE)

The mean square error (MSE) of an estimator $\hat{\theta}$ For estimating θ is:-

$$MSE_{\theta}(\hat{\theta}) = E(\theta-\theta)^2 = var(\hat{\theta}) + (Bias_{\theta}(\hat{\theta}))^2$$

* bias $= E(\hat{\theta}) - \theta = \theta - \theta =$ zero

* if $\hat{\theta}$ is unbiased then $MSE(\hat{\theta}) = var(\hat{\theta})$

* if $MSE(\hat{\theta})$ is smaller, $\hat{\theta}$ is a better estimator of θ.

* For two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ

$\rightarrow MSE_{\theta}(\hat{\theta}_1) < MSE_{\theta}(\hat{\theta}_2)$, $\theta \in \Omega$

$\rightarrow \hat{\theta}_1$ is better estimator of θ.

Ex:- let we have two estimators For Parameter θ, which They are $\hat{\theta}_1$ and $\hat{\theta}_2$ where $MSE(\hat{\theta}_1) = 0.758$ and $MSE(\hat{\theta}_2) = 0.427$ which estimator is better?

Solution:- Since $MSE(\hat{\theta}_1) = 0.758$

$MSE(\hat{\theta}_2) = 0.427$

$\rightarrow MSE(\hat{\theta}_2) < MSE(\hat{\theta}_1)$

$\therefore \hat{\theta}_2$ is better estimator for θ.