Theorem (1):

1. $\lim\limits_{\theta\to\infty} \frac{\sin(\theta)}{(\theta)} = 0$
2. $\lim\limits_{\theta\to\infty} \frac{\cos(\theta)}{(\theta)} = 0$
3. $\lim\limits_{\theta\to\infty} \frac{1}{\theta} = 0$
4. $\lim\limits_{\theta\to 0} \frac{1}{\theta} = \infty$

Theorem (2):

1- $\lim\limits_{x\to\theta} sin(x) = \sin(\theta)$

2- $\lim\limits_{x\to\theta} cos(x) = \cos(\theta)$

3- $\lim\limits_{x\to 0} \frac{\sin(x)}{(x)} = 1$

4- $\lim\limits_{\theta\to 0} \frac{\tan(\theta)}{(\theta)} = 0$

Theorem (3):

If f and g are two functions with $\lim\limits_{x\to x_0} f(x) = L_1$ and $\lim\limits_{x\to x_0} g(x) = L_2$ then,

1- $\lim\limits_{x\to x_0} k = k,$ (k is any constant).

2- $\lim\limits_{x\to x_0} x = x_0$

3- $\lim\limits_{x\to x_0} c\, f(x) = cL_1$

4- $\lim_{x\to x_0}(f(x)\pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = L_1 \pm L_2$

5- $\lim_{x\to x_0}(f(x).g(x)) = \lim_{x\to x_0} f(x).\lim_{x\to x_0} g(x) = L_1.L_2$

6- $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2},$ such that $L_2 \neq 0.$

Theorem (4):

1- if $\lim_{x\to x_0} f(x) = L$, then

$$\lim_{x\to x_0}[f(x)]^n = [\lim_{x\to x_0} f(x)]^n = [L]^n$$

2- $\lim_{x\to x_0}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)} = \sqrt[n]{L}$, $L \geq O$, n is even.

Theorem (5):

If $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n$
is any polynomial function, then

$$\lim_{x\to x_0} f(x) = f(x_0) = a_0 + a_1x_0 + a_2{x^2}_0 + a_3{x^3}_0 + a_4{x^4}_0 + \ldots + a_n{x^n}_0$$

Examples:

1- Prove that $\lim_{h \to 0} \frac{2\sin(h).\cos(h)}{h} = 2$

Solution:

To prove $\lim_{h \to 0} \frac{2\sin(h).\cos(h)}{h} = 2$

From

$$\sin(2h) = 2\sin(h).\cos(h)$$

So $$\lim_{h \to 0} \frac{2\sin(h).\cos(h)}{h} = \lim_{h \to 0} \frac{\sin(2h)}{h}$$

$$\lim_{h \to 0} \frac{\sin(2h)}{h} = 2\lim_{h \to 0} \frac{\sin(2h)}{2h}$$

بالضرب
$\frac{2}{2}$ ×

By using theorem (2),

$$\lim_{h \to 0} \frac{2\sin(h).\cos(h)}{h} = 2$$

2- Find $\lim_{x \to \infty} \frac{x+\sin(x)}{x+\cos(x)}$

Solution:

$$\lim_{x \to \infty} \frac{x+\sin(x)}{x+\cos(x)} = \lim_{x \to \infty} \frac{\frac{x}{x}+\frac{\sin(x)}{x}}{\frac{x}{x}+\frac{\cos(x)}{x}}$$

بالقسمة
على x

$$= \lim_{x \to \infty} \frac{1 + \frac{\sin(x)}{x}}{1 + \frac{\cos(x)}{x}}$$

by using theorem (1).

$$\lim_{x \to \infty} \frac{1 + \frac{\sin(x)}{x}}{1 + \frac{\cos(x)}{x}} = \lim_{x \to \infty} \frac{1+0}{1+0} = 1$$

3- Find $\lim_{x \to 0} \frac{sin^2 x}{x}$

Solution:

$$\lim_{x \to 0} \frac{sin^2 x}{x} = \lim_{x \to 0} \frac{\sin(x) . \sin(x)}{x}$$

$$= \lim_{x \to 0} [\sin(x) . \frac{\sin(x)}{x}]$$

$$= \lim_{x \to 0} \sin(x) . \lim_{x \to 0} \frac{\sin(x)}{x}$$

By using theorem (2)

$$= \lim_{x \to 0} \sin(x) . 1 = 0$$

4- Find $\lim\limits_{x \to 0} \frac{\sin(4x)}{x}$

Solution:

بالضرب $\frac{4}{4} \times$

$$\lim_{x \to 0} \frac{sin4x}{x} = \lim_{x \to 0} \frac{4\sin(4x)}{4x}$$

$$= 4 \lim_{x \to 0} \frac{\sin(4x)}{4x}$$

By using theorem (2).

$$= 4.(1) = 4$$

5- Find $\lim\limits_{x \to \frac{\pi}{2}} \sin(x) = 0$

Solution:

بالتعويض المباشر

$$\lim_{x \to \frac{\pi}{2}} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

6- Find $\lim\limits_{x \to 0} \frac{\sin(3x)}{\sin(5x)}$

Solution:

بالضرب مرة $\frac{3x}{3x} \times$

ومرة ثانية $\frac{5x}{5x} \times$

$$\lim_{x \to 0} \frac{sin3x}{sin5x} = \lim_{x \to 0} \frac{\frac{(3x).\sin(3x)}{3x}}{\frac{(5x).\sin(5x)}{5x}}$$

$$= \frac{\lim_{x\to 0}(3x).\lim_{x\to 0}\left(\frac{\sin(5x)}{5x}\right)}{\lim_{x\to 0}(5x).\lim_{x\to 0}\left(\frac{\sin(5x)}{5x}\right)}$$

By using theorem (2).

$$= \frac{\lim_{x\to 0}(3x).1}{\lim_{x\to 0}(5x).1}$$

$$= \lim_{x\to 0}\left(\frac{3x}{5x}\right) = \lim_{x\to 0}\left(\frac{3}{5}\right) = \frac{3}{5}$$

Remark:

1- If the value of $f(x)$ approaches to L_1 as x Approaches to x_0 from the right side we write $\lim_{x\to x_0^+} f(x) = L_1$.

2- If the value of $f(x)$ approaches to L_2 as x Approaches to x_0 from the left side we write $\lim_{x\to x_0^-} f(x) = L_2$.

3- If the limit from the left side equals the limit from the right side (i.e if$L_1 = L_2 = K$)

Then $\lim_{x \to x_0} f(x)$ exist and $\lim_{x \to x_0} f(x) = k$

But if $L_1 \neq L_2$,Then the $\lim_{x \to x_0} f(x)$ does not exist.

Example:

Find $\lim_{x \to 3} f(x)$ if it exists where

$$f(x) = \begin{bmatrix} 4+x & ,x \geq 3 \\ x-4 & ,x < 3 \end{bmatrix}$$

Solution:

Right side

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (4 + x) = 4 + 3 = 7 = L_1$$

Left side

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x\_4) = 3_4 = -1 = L_2$$

Now, since

$\lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$, then $\lim_{x \to 3} f(x)$ does not exist.

Example:

find $\lim_{x\to 2} f(x)$ if it exists

$$f(x) = \begin{bmatrix} x+4 & & if & x\geq 2 \\ x^2+2 & & if & x<2 \end{bmatrix}$$

Solution:

Right side

$$\lim_{2^+} f(x) = \lim_{2^+}(x+4) = 2+4 = 6$$

Left side

$$\lim_{2^-} f(x) = \lim_{2^-} x^2 + 2 = 2^2 + 2 = 6$$

Since $\lim_{2^+} f(x) = \lim_{2^-} f(x) = 6$

Thus $\lim_{x\to 2} f(x) = 6$