

The derivative of the trigonometric functions

Theorem:

Let u be function then.

$$1- d(\sin u) = \cos u . du$$

$$2- d(\cos u) = -\sin u . du$$

$$3- d(\tan u) = \sec^2 u . du$$

$$4- d(\cot u) = -\csc^2 u . du$$

$$5- d(\sec u) = \sec u . \tan u . du$$

$$6- d(\csc u) = -\csc u . \cot u . du$$

Example:

Find $y' = \frac{dy}{dx}$ of the following functions.

$$1- f(x) = x^3 + \sin 2x$$

$$2- y = \frac{\sin x}{(\tan x)+1}$$

Sol/

$$1- y' = \frac{dy}{dx} = 3x^2 + [\cos 2x . (2)]$$

$$= 3x^2 + 2 \cos 2x$$

$$\begin{aligned} 2- y' &= \frac{(\tan x+1)[\cos x . 1] - [\sin x(\sec^2 x+0)]}{(\tan x+1)^2} \\ &= \frac{\tan x \cos x + \cos x - \sin x \sec^2 x}{(\tan x + 1)^2} \end{aligned}$$

$$\frac{\frac{\sin x}{\cos x} \cos x + \cos x - \sin x \sec^2 x}{(\tan x + 1)^2}$$

$$\frac{\sin x + \cos x - \sin x \sec^2 x}{(\tan x + 1)^2}$$

Home work:

Find $y' = \frac{dy}{dx}$ of the following functions.

1- $y = 4 \sec x + \tan x$

3- $y = \sqrt{x} + \cot x$

2- $y = (\csc x) \cdot (x + 1)$

4- $y = \frac{x^2+2}{\cos x^2}$

Example:

Find y' for each of the following functions.

1- $y = \sin(x^2 + 2x + 1)$

Sol/ $y' = \cos(x^2 + 2x + 1) [2x + 2 + 0]$

$$= (2x + 2) \cdot \cos(x^2 + 2x + 1)$$

2- $y = \tan(x + \cos x)$

Sol/

Let $u = x + \cos x$

$$y = \tan u$$

$$y' = \sec^2 u \cdot du$$

Now, $u = x + \cos x$

$$du = 1 + (-\sin x (1))$$

$$= 1 - \sin x$$

$$\Rightarrow y' = \sec^2(x + \cos x) \cdot (1 - \sin x)$$

$$= \sec^2(x + \cos x) - \sin x \cdot \sec^2(x + \cos x)$$

Home work: Find y' for each of the following functions.

1- $y = \cot(\sec x)$

2- $y = \frac{\sin(4x)}{x^2 + \cos x}$

3- $y = \frac{\sqrt{x}}{\sin x}$

4- $y = \frac{\tan(\sin x)}{\cot(\cos x)}$

Derivative of composite functions

Chain Rule :

Let $y = f(u)$, $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

قاعدة السلسلة

لدينا دالتين الاولى هي y معرفة بدلالة u والثانية هي u معرفة بدلالة x والمطلوب هو مشتقة ال y بدلالة ال x وعليه نطبق قاعدة السلسلة

Example:

Let $y = 5u^2 + 3u$, $u = \sin x$ find $\frac{dy}{dx}$

Sol/

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now $\frac{dy}{du} = 10u + 3$ and $\frac{du}{dx} = \cos x$

Thus $\frac{dy}{dx} = (10u + 3)(\cos x)$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= (10(\sin x) + 3)(\cos x) \\ &= 10 \sin x \cos x + 3 \cos x \end{aligned}$$

نعوض بدل كل u بما يساويها لكي نحصل على مشتقة y بدلالة x

Example:

Find $\frac{dy}{dx}$ for each of the following functions

1- $y = \sin(\tan x)$

Sol/

Let $u = \tan x \Rightarrow y = \sin u$

$$\frac{dy}{du} = \cos u \quad \text{and} \quad \frac{du}{dx} = \sec^2 x$$

Thus by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \sec^2 x = \cos(\tan x) \cdot \sec^2 x$$

$$2- y = (x^2 + 5x)^{10}$$

Sol/

$$\text{Let } u = x^2 + 5x \Rightarrow y = u^{10}$$

$$\frac{dy}{du} = 10 \cdot u^9 \quad \text{and} \quad \frac{du}{dx} = 2x + 5$$

by chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = (10 \cdot u^9)(2x + 5) \\ &= 10(x^2 + 5x)^9(2x + 5) \end{aligned}$$

$$3- y = \cos(\sec x)$$

H.W

Second Order Derivative and Derivatives of Higher Order

The derivative $y' = \frac{dy}{dx}$ is the first derivative of $y = f(x)$ with respect to x . The first derivative is itself a function of x and may be differentiable.

Thus, we can derivative it

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 f}{d^2 x} = \frac{d^2 y}{d^2 x}$$

Which is called second derivative of $y = f(x)$ with respect to x .

In general

$$y''' = \frac{d^3 y}{d^3 x} = \frac{d}{dx} \left(\frac{d^2 y}{d^2 x} \right) \text{ is the third derivative.}$$

$$y'''' = y^{(4)} = \frac{d^4 y}{d^4 x} = \frac{d}{dx} \left(\frac{d^3 y}{d^3 x} \right) \text{ is the fourth derivative.}$$

⋮

⋮

$$y^{(n)} = \frac{d^n y}{d^n x} = \frac{d}{dx} \left(\frac{d^{n-1} y}{d^{n-1} x} \right) \text{ is } n \text{ the derivative.}$$

Example:

Find $\dot{y}$ y'' y''' $y^{(4)}$ of $y = \sin(x^3)$

Sol/

$$\dot{y} = \cos(x^3) \cdot 3x^2 = 3x^2 \cos x^3$$

$$y'' = (3x^2)(-\sin x^3 (3x^2)) + (\cos x^3)(6x)$$

$$= -9x^4 \sin x^3 + 6x \cos x^3$$

$$y''' = [-9x^4(3x^2 \cos(x^3)) + (\sin(x^3))(-36x^3)] \\ + [6x(-\sin x^3 (3x^2)) + 6 \cos x^3]$$

$$y^{(4)} \rightarrow H.M$$

Example:

Let $y = x^4 + 3x^2 - 5x + 2$ $y'''(\frac{1}{6})$ and $y^{(4)}(\frac{1}{6})$

Sol/

$$\dot{y} = 4x^3 + 6x - 5$$

$$y'' = 12x^2 + 6$$

$$y''' = 24x \quad y'''(\frac{1}{6}) = (24)(\frac{1}{6}) = 4$$

$$y^{(4)} = 24 \rightarrow y^{(4)}(\frac{1}{6}) = 24$$

Home work:

1. let $f(x) = x \sin x$ find $f'(\frac{\pi}{4})$
2. show that $y'' + 4y = 0$, when $y = 3 \sin(2x + 3)$
3. show that $y''' + y'' + \dot{y} + y = 0$ when $y = \sin x + 2 \cos x$
4. Find $\dot{y}$ y'' y''' for the following functions
 - I. $y = x^4 + 1$
 - II. $y = \cot x$
 - III. $y = 5 \sec x$
 - IV. $y = \sqrt{\sin x}$
 - V. $\cos(x + 2)^2$
 - VI. $y = \frac{1}{2} \tan x \sin 2x$