Chapter 5: Angular Momentum

5.1. Definition:

In classical mechanics, the angular momentum is defined as:

$$\vec{L} = \vec{r} \times \vec{p} \quad .....(5\text{-}1\text{-}1)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 \quad .....(5\text{-}1\text{-}2)$$

$$\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned} \quad .....(5\text{-}1\text{-}3)$$

In quantum mechanics

$$\begin{aligned} \hat{L}_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \\ \hat{L}_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \\ \hat{L}_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \end{aligned} \quad ....(5\text{-}1\text{-}4)$$

According to the following commutation relations, the components of angular momentum are not commute with each other ($\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ are incompatible observables).

$$\begin{aligned} \left[\hat{L}_x, \hat{L}_y\right] &= i\hbar\hat{L}_z \quad , \quad \left[\hat{L}_y, \hat{L}_x\right] = -i\hbar\hat{L}_z \\ \left[\hat{L}_y, \hat{L}_z\right] &= i\hbar\hat{L}_x \quad , \quad \left[\hat{L}_z, \hat{L}_y\right] = -i\hbar\hat{L}_x \\ \left[\hat{L}_z, \hat{L}_x\right] &= i\hbar\hat{L}_y \quad , \quad \left[\hat{L}_x, \hat{L}_z\right] = -i\hbar\hat{L}_y \end{aligned} \quad (5\text{-}1\text{-}5)$$

Ex. Prove $\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$.

Soln.

$$\left[\hat{L}_x,\hat{L}_y\right]=\hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x=(yp_z-zp_y)(zp_x-xp_z)-(zp_x-xp_z)(yp_z-zp_y)$$
$$=yp_zzp_x-yp_zxp_z-zp_yzp_x+zp_yxp_z-zp_xyp_z+zp_xzp_y+xp_zyp_z+xp_zzp_y$$
$$=(yp_zzp_x-zp_xyp_z)+(xp_zyp_z-yp_zxp_z)+(zp_yxp_z-xp_zzp_y)+(zp_xzp_y-zp_yzp_x)$$

$$(yp_zzp_x-zp_xyp_z)=yp_x(p_zz-zp_z)=-i\hbar yp_x$$
$$(xp_zyp_z-yp_zxp_z)=0$$
$$(zp_yxp_z-xp_zzp_y)=p_yx(zp_z-p_zz)=i\hbar p_yx$$
$$(zp_xzp_y-zp_yzp_x)=0$$

$$\therefore\quad\left[\hat{L}_x,\hat{L}_y\right]=i\hbar(p_yx-p_xy)=i\hbar\hat{L}_z$$

Where $[x,p_x]=[y,p_y]=[z,p_z]=i\hbar$ and $[x,p_y]=[y,p_x]=[z,p_x]=0$ → $[r_i,p_j]=i\hbar\delta_{ij}$

H.W. Prove $[\hat{x},\hat{p}_x]=i\hbar$.

H.W. Prove $\left[\hat{L}_z,\hat{z}\right]=0$.

H.W. Prove $\left[\hat{L}_z,\hat{x}\right]=i\hbar y$

H.W. Prove $\left[\hat{L}_z,\hat{p}_z\right]=0$.

H.W. Prove that $\left[\hat{L}_z,\hat{p}_x\right]=i\hbar\hat{p}_y$.

Ex. Prove $\left[\hat{L}_x,\hat{L}^2\right]=0$.

Soln.

$$\left[\hat{L}_x,\hat{L}^2\right]=\left[L_x,L_x^2\right]+\left[L_x,L_y^2\right]+\left[L_x,L_z^2\right]$$

$$\left[L_x,L_x^2\right]=L_xL_xL_x-L_xL_xL_x=0$$

$$\left[L_x,L_y^2\right]=L_xL_y^2-L_y^2L_x=L_xL_yL_y-L_y^2L_x+L_yL_xL_y-L_yL_xL_y=\left(L_xL_yL_y-L_yL_xL_y\right)+L_yL_xL_y-L_y^2L_x$$
$$=\left(L_xL_yL_y-L_yL_xL_y\right)+\left(L_yL_xL_y-L_yL_yL_x\right)=(L_xL_y-L_yL_x)L_y+L_y(L_xL_y-L_yL_x)$$
$$=\left[L_x,L_y\right]L_y+L_y\left[L_x,L_y\right]=i\hbar L_zL_y+i\hbar L_yL_z$$

$$\left[L_x, L_z^2\right] = L_xL_zL_z - L_zL_zL_x = (L_xL_z - L_zL_x)L_z + L_z(L_xL_z - L_zL_x)$$
$$= \left[L_x, L_z\right]L_z + L_z\left[L_x, L_z\right] = -i\hbar L_yL_z - i\hbar L_zL_y$$

$$\left[\hat{L}_x, \hat{L}^2\right] = 0 + i\hbar L_zL_y + i\hbar L_yL_z - i\hbar L_yL_z - i\hbar L_zL_y = 0$$

5.2. Ladder operators (raising and lowering operator)

The raising $\hat{L}_+$ and lowering $\hat{L}_-$ operators are defined as

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$
$$\hat{L}_- = \hat{L}_x - i\hat{L}_y \qquad \text{.....(5-2-1)}$$

Both $\hat{L}_+$ and $\hat{L}_-$ are non-Hermitian operators satisfy the following commutation relations:

$$[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z \qquad \text{.....(5-2-2)}$$

$$[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm \qquad \text{.....(5-2-3)}$$

$$[\hat{L}^2, \hat{L}_\pm] = 0 \qquad \text{.....(5-2-4)}$$

H.W. Prove $[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$.

Ex. Prove $[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm$.

Soln.

$$[L_z, L_\pm] = L_zL_\pm - L_\pm L_z = L_z(L_x \pm iL_y) - (L_x \pm iL_y)L_z$$
$$[L_z, L_\pm] = L_zL_x \pm iL_ZL_y - L_xL_z \mp iL_yL_z = \left[L_z, L_x\right] \pm i\left[L_z, L_y\right]$$
$$[L_z, L_\pm] = i\hbar L_y \pm i(-i\hbar L_x) = i\hbar L_y \pm \hbar L_x = \pm\hbar(L_x \pm iL_y) = \pm\hbar L_\pm$$

H.W. Prove $[\hat{L}^2, \hat{L}_\pm] = 0$.

Since, the operator $\hat{L}^2$ commute with each component of angular momentum ($\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$), i.e. $\left[\hat{L}_x, \hat{L}^2\right] = \left[\hat{L}_y, \hat{L}^2\right] = \left[\hat{L}_z, \hat{L}^2\right] = 0$. Therefore, it is possible to find simultaneous eigenfunction of $\hat{L}^2$ and any component of $\hat{L}$ (By convention we choose $\hat{L}_z$).

From B.Sc. course of QM, we have found that the eigenfunction of both $\hat{L}^2$ and $\hat{L}_z$ is the spherical harmonic function $Y_\ell^m(\theta,\phi) = |\ell m\rangle$, with

$$\hat{L}^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle \qquad \text{.....(5-2-5)}$$

$$\hat{L}_z|\ell m\rangle = m\hbar|\ell m\rangle \qquad \text{.....(5-2-6)}$$

Where ℓ and m are real numbers (since $\hat{L}^2$ and $\hat{L}_z$ are Hermitian operators).

$\ell = 0, 1, 2, ...$ and $-\ell \le m \le \ell$

To demonstrate the action of $\hat{L}_\pm$ on $|\ell m\rangle$

Let apply the operators $\hat{L}_z\hat{L}_+$ on $|\ell m\rangle$, i.e. $\hat{L}_z\hat{L}_+|\ell m\rangle$

From $[\hat{L}_z, \hat{L}_+] = \hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z = \hbar\hat{L}_+$

$$\hat{L}_z\hat{L}_+ = \hat{L}_+\hat{L}_z + \hbar\hat{L}_+ \qquad \text{.....(5-2-7)}$$

$$\hat{L}_z\hat{L}_+|\ell m\rangle = (\hat{L}_+\hat{L}_z + \hbar\hat{L}_+)|\ell m\rangle = \hbar(m+1)\hat{L}_+|\ell m\rangle \qquad \text{.....(5-2-8)}$$

If we consider $\hat{L}_+|\ell m\rangle$ as a whole to be just another eigen function of $\hat{L}_z$, then this result indicates that $\hat{L}_+$ must operate on $|\ell m\rangle$ to give $|\ell m+1\rangle$, i.e.

$$\hat{L}_+|\ell m\rangle = c_+|\ell m+1\rangle \qquad \text{.....(5-2-9)}$$

Similarly, we can get an equivalent result for $\hat{L}_-$ i.e.

$$\hat{L}_-|\ell m\rangle = c_-|\ell m-1\rangle \qquad \text{.....(5-2-10)}$$

In general

$$\hat{L}_\pm|\ell m\rangle = c_\pm|\ell m\pm 1\rangle \qquad \text{.....(5-2-11)}$$

To find c_+

$$\langle \ell' m'|\hat{L}_+^\dagger\hat{L}_+|\ell m\rangle = \langle \ell' m'|\hat{L}_-\hat{L}_+|\ell m\rangle$$

$$\langle \ell' m'+1|c_+^* c_+|\ell m+1\rangle = \langle \ell' m'|\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z|\ell m\rangle$$

$$|c_+|^2\delta_{\ell'\ell}\delta_{m'm}=\langle\ell'm'|\hat{L}^2-\hat{L}_z^2-\hbar\hat{L}_z|\ell m\rangle$$

$$|c_+|^2\delta_{\ell'\ell}\delta_{m'm}=\langle\ell'm'|\ell(\ell+1)\hbar^2-m^2\hbar^2-m\hbar^2|\ell m\rangle$$

$$|c_+|^2\delta_{\ell'\ell}\delta_{m'm}=\left(\ell(\ell+1)\hbar^2-m^2\hbar^2-m\hbar^2\right)\delta_{\ell'\ell}\delta_{m'm}$$

$$c_+=\hbar\sqrt{\ell(\ell+1)-m^2-m}=\hbar\sqrt{\ell(\ell+1)-m(m+1)} \quad \text{.....(5-2-12)}$$

Similarly, one can find c_-.

H.W. If $L_-|\ell m\rangle=c_-|\ell m-1\rangle$, find c_-.

To demonstrate the action of $\hat{L}_x$ and $\hat{L}_y$ on $|\ell m\rangle$

$\hat{L}_x$ and $\hat{L}_y$ operators can be reformulated as:

$$\hat{L}_x=\frac{1}{2}\left(\hat{L}_++\hat{L}_-\right) \quad \text{(5-2-13)}$$

$$\hat{L}_y=\frac{-i}{2}\left(\hat{L}_+-\hat{L}_-\right) \quad \text{(5-2-14)}$$

$$\hat{L}_x|\ell m\rangle=\frac{1}{2}\left(\hat{L}_++\hat{L}_-\right)|\ell m\rangle=\frac{1}{2}\left(c_+|\ell m+1\rangle+c_-|\ell m-1\rangle\right) \quad \text{.....(5-2-15)}$$

$$\hat{L}_y|\ell m\rangle=\frac{-i}{2}\left(\hat{L}_+-\hat{L}_-\right)|\ell m\rangle=\frac{-i}{2}\left(c_+|\ell m+1\rangle-c_-|\ell m-1\rangle\right) \quad \text{.....(5-2-16)}$$

5.3. Matrix representation of angular momentum operators

$\hat{L}^2$, $\hat{L}_z$, $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_+$ and $\hat{L}_-$ operators can be represented in matrix form as follows**:**

i) $\hat{L}^2|\ell m\rangle=\hbar^2\ell(\ell+1)|\ell m\rangle$

$$\langle\ell'm'|\hat{L}^2|\ell m\rangle=\hbar^2\ell(\ell+1)\langle\ell'm'|\ell m\rangle$$
$$\langle\ell'm'|\hat{L}^2|\ell m\rangle=\hbar^2\ell(\ell+1)\delta_{\ell'\ell}\delta_{m'm}$$

In matrix form, for $\ell=1$ → $m=\pm1,0$

$$L^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad .....(5\text{-}3\text{-}1)$$

ii) $\hat{L}_z |\ell m\rangle = m\hbar |\ell m\rangle$

$$\langle \ell' m' | \hat{L}_z | \ell m\rangle = m\hbar \langle \ell' m' | \ell m\rangle$$

$$\langle \ell' m' | \hat{L}_z | \ell m\rangle = m\hbar \delta_{\ell'\ell} \delta_{m'm}$$

In matrix form, for $\ell = 1$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad .....(5\text{-}3\text{-}2)$$

Ex. Express the lowering and raising angular momentum operator in matrix form for $\ell = 1$.

Soln.

i) $\hat{L}_+ |\ell m\rangle = c_+ |\ell m+1\rangle$

$$\langle \ell' m' | \hat{L}_+ | \ell m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m+1)}\, \delta_{\ell'\ell} \delta_{m',m+1}$$

In matrix form, for $\ell = 1$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

ii) $\hat{L}_- |\ell m\rangle = c_- |\ell m-1\rangle$

$$\langle \ell' m' | \hat{L}_- | \ell m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m-1)}\, \delta_{\ell'\ell} \delta_{m',m-1}$$

In matrix form, for $\ell = 1$ → $L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

H.W. Express $\hat{L}_x$ and $\hat{L}_y$ operators in matrix form for $\ell = 1$.