

2- The Definite integral التكامل المحدد

Theorem:

If f is continuous on $[a, b]$ and F is any anti_derivation of f $[a, b]$ then

$$\int_a^b f(x) dx = F(b) - F(a)$$

It is called the definite integral.

Example:

Evaluate the integral $\int_1^2 (x^4 + 3x^2 - x + 1) dx$

Solution:

$$\begin{aligned}\int_1^2 (x^4 + 3x^2 - x + 1) dx &= \left[\frac{x^5}{5} + \frac{3x^3}{3} - \frac{x^2}{2} + x \right]_1^2 \\&= \left[\frac{(2)^5}{5} + \frac{3(2)^3}{3} - \frac{(2)^2}{2} + 2 \right] - \left[\frac{(1)^5}{5} + \frac{3(1)^3}{3} - \frac{(1)^2}{2} + 1 \right] \\&= \left[\frac{32}{5} + 8 - 2 + 2 \right] - \left[\frac{1}{5} + 1 - 1 + 1 \right] \\&= \frac{31}{5} + 7 \\&= \frac{66}{5}\end{aligned}$$

Example:

Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \cos x \, dx$$

Solution:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos x \, dx &= [\sin x]_0^{\frac{\pi}{2}} \\ &= \sin \frac{\pi}{2} - \sin 0 \\ &= 1 \end{aligned}$$

Properties of definite integral

خواص التكامل المحدد

1- If a is in the domain of f we have

$$\int_a^a f(x) dx = 0$$

$$2- \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$3- \int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$4- \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

5- If $f(x)$ is continuous on $[a, c]$ such that $a < b < c$ then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Example:

Evaluate the following integrals

1- $\int_0^3 (x^2 + 2x)dx$

$$\begin{aligned}\int_0^3 (x^2 + 2x)dx &= \left[\frac{x^3}{3} + 2\frac{x^2}{2} \right]_0^3 \\ &= \left[\left(\frac{(3)^3}{3} + 2\frac{(3)^2}{2} \right) - \left(\frac{(0)^3}{3} + 2\frac{(0)^2}{2} \right) \right] \\ &= 18\end{aligned}$$

2- $\int_{-1}^0 (x + 1)dx$

$$\int_{-1}^0 (x + 1)dx = \left[\frac{x^2}{2} + x \right]_{-1}^0$$

$$= (0) - \left(\frac{1}{2} - 1 \right)$$

$$= \frac{1}{2}$$

$$3- \int_4^5 \frac{dx}{(x-3)}$$

$$\int_4^5 \frac{dx}{(x-3)} = [\ln (x-3)]_4^5$$

$$= \ln (5-3) - \ln (4-3)$$

$$= \ln (2) - \ln (1)$$

$$= \ln (2)$$

$$4- \text{Let } f(x) = \begin{cases} 3x-2, & x < 2 \\ x^2, & x \geq 2 \end{cases} \quad \text{find } \int_0^6 f(x)dx$$

Solution:

$$\int_0^6 f(x)dx = \int_0^2 (3x-2)dx + \int_2^6 (x^2)f(x)dx$$

$$= \left[\frac{3x^2}{2} - 2x \right]_0^2 + \left[\frac{x^3}{3} \right]_2^6$$

$$= \left(\frac{3(2)^2}{2} - 4 \right) + \left[\left(\frac{(6)^3}{3} \right) - \left(\frac{(2)^3}{3} \right) \right]$$

$$=$$

تطبيقات التكامل المحدد Application of the Definite Integral

المساحة تحت المنحني The area under curve

We can find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a,b]$ by the following formula

$$\text{Area} = \left| \int_a^b f(x) dx \right|$$

م | تم وضع علامة المطلق حول التكامل لان المساحة موجبة دائما .

Example:

Find the Area between the curve $y = x^2 - 2x + 3$ and the x-axis from $x = 0$ to $x = 3$

solution:

$$\begin{aligned} \text{Area} &= \left| \int_a^b f(x) dx \right| \\ &= \left| \int_0^3 (x^2 - 2x + 3) dx \right| \\ &= \left| \left[\frac{x^3}{3} - \frac{2x^2}{2} + 3x \right]_0^3 \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \left(\frac{(3)^3}{3} - \frac{2(3)^2}{2} + 3(2) \right) - \left(\frac{(0)^3}{3} - \frac{2(0)^2}{2} + 3(0) \right) \right| \\
&= \left| \left(\frac{27}{3} - 9 + 9 \right) - (0) \right| \\
&= 9
\end{aligned}$$

Example:

Find the Area between the curve $y = 4 - x^2$ and the x-axis.

م | في هذا المثال اعطى منحنى الدالة فقط ولم يعطى حدود التكامل وعليه يجب ايجاد حدود التكامل اولاً .

ولايجاد حدود التكامل نساوي الدالة بالصفر ونجد قيم x التي تمثل حدود التكامل .

Solution:

$$y = 4 - x^2$$

$$\Rightarrow 4 - x^2 = 0$$

$$\Rightarrow (2 - x)(2 + x) = 0$$

$$\Rightarrow \text{either } x = -2 \text{ or } x = 2$$

الآن اصبحت حدود التكامل $a=-2$ و $b=2$

$$\begin{aligned} \text{Area} &= \left| \int_a^b f(x) dx \right| \\ &= \left| \int_{-2}^2 (4 - x^2) dx \right| \\ &= \left| \left[4x - \frac{x^3}{3} \right]_{-2}^2 \right| \\ &= \left| \left(4(2) - \frac{(2)^3}{3} \right) - \left(4(-2) - \frac{(-2)^3}{3} \right) \right| \\ &= \frac{32}{3} \end{aligned}$$

Example:

Find the total Area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:

$$x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0$$

$$x(x - 2)(x + 2) = 0$$

either $x = 0$ or $x = 2$, $x = -2$

نلاحظ هنا اصبحت لدينا ثلاثة حدود

$$\therefore \text{Area} = |A_1| + |A_2|$$

$$= \left| \int_{-2}^0 (x^3 - 4x) dx \right| + \left| \int_0^2 (x^3 - 4x) dx \right|$$

$$|A_1| = \left| \int_{-2}^0 (x^3 - 4x) dx \right| = \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_{-2}^0 = \left| 0 - \left(\frac{16}{4} - 8 \right) \right|$$
$$= 4$$

$$|A_2| = \left| \int_0^2 (x^3 - 4x) dx \right| = \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_0^2 = \left| \left[\frac{16}{4} - 8 \right] - [0] \right|$$
$$= 4$$

$$\text{Thus Area} = |A_1| + |A_2| = 4 + 4 = 8$$

Example:

Find the total Area of the region between the curves

$y = x^3 - x^2 - 6x$ and the x-axis over the interval $[-2, 2]$.

Solution:

$$x^3 - x^2 - 6x = 0 \Rightarrow x(x^2 - x - 6) = 0$$

$$\Rightarrow x(x - 3)(x + 2) = 0$$

Either $x = 0$ or $x = 3$, $x = -2$

نلاحظ هناك ثلاث قيم لل

نحدد القيم التي داخل الفترة المعطاة في السؤال فقط

وعليه 0, -2 هي داخل الفترة ولكن 3 خارج الفترة وعليه تهمل في الحل

$$\text{Area} = |A_1| + |A_2|$$

$$\text{Area} = \left| \int_{-2}^0 f(x) dx \right| + \text{Area} = \left| \int_0^2 f(x) dx \right|$$

$$|A_1| = \left| \int_{-2}^0 (x^3 - x^2 - 6x) \right| = \left| \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2} \right]_{-2}^0 \right|$$

$$= \left| [0] - \left[\frac{16}{4} - \frac{-8}{3} - \frac{24}{2} \right] \right| = \frac{16}{3}$$

$$|A_2| = \left| \int_0^2 (x^3 - x^2 - 6x) \right| = \left| \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2} \right]_0^2 \right|$$

$$= \left| \left[\frac{16}{4} - \frac{8}{3} - \frac{24}{2} \right] - [0] \right|$$

$$= \frac{32}{3}$$

$$\text{Thus Area} = |A_1| + |A_2| = \frac{16}{3} + \frac{32}{3} = 16$$

Double integral

التكامل المضاعف

The properties of double integrals:

Double integrals over region R of continuous functions have the following properties:

$$1- \iint_R k(f(x, y)) dx dy = k \iint_R (f(x, y)) dx dy$$

نلاحظ هنا $f(x, y)$ هي دالة معرفة بمتغيرين هما x, y

$$2- \iint_R [f(x, y) \pm g(x, y)] dx dy = \iint_R (f(x, y)) dx dy \pm \iint_R (g(x, y)) dx dy$$

3- If $R = R_1 \cup R_2$ then

$$\iint_R (f(x, y)) dx dy = \iint_{R_1} f(x, y) dx dy + \iint_{R_2} f(x, y) dx dy$$

Theorem: (Fubini's)(first order)

If $f(x, y)$ is continuous function on the rectangular reigh

$R: a \leq x \leq b, c \leq y \leq d$ then

$$\iint_R f(x, y) dx dy = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Example:

Calculate $\int_2^3 \int_0^1 (x + 3y) dx dy$

Solution:

$$\int_2^3 \left[\left[\frac{x^2}{2} + 3yx \right]_0^1 \right] dy$$

$$= \int_2^3 \left[\left(\frac{1^2}{2} + 3y(1) \right) - (0) \right] dy$$

$$= \int_2^3 \left(\frac{1}{2} + 3y \right) dy$$

$$= \left[\left(\frac{1}{2}y + 3 \frac{y^2}{2} \right) \right]_2^3$$

البداية مع x يعني نكامل بالنسبة
لل x وال y يعتبر ثابت

$$= \left(\frac{1}{2}(3) + 3 \frac{(3)^2}{2} \right) - \left(\frac{1}{2}(2) + 3 \frac{(2)^2}{2} \right)$$

$$= \left(\frac{30}{2} \right) - (7)$$

$$= 8$$

Example:

Calculate $\int_0^1 \int_2^3 (x + 3y) dy dx$

Solution:

$$\int_0^1 \int_2^3 (x + 3y) dy dx = \int_0^1 \left[xy + 3 \frac{y^2}{2} \right]_2^3 dx$$

$$\int_0^1 \left(x(3) + 3 \frac{(3)^2}{2} \right) - \left(x(2) + 3 \frac{(2)^2}{2} \right) dx$$

$$\int_0^1 \left(x + \frac{15}{2} \right) dx$$

$$= \left[\frac{x^2}{2} + \frac{15}{2} x \right]_0^1$$

$$= \left(\frac{(1)^2}{2} + \frac{15}{2} (1) \right) - (0)$$

$$= \frac{1}{2} + \frac{15}{2}$$

$$= 8$$

Home work:

Calculate each of the following double integrals

1- $\int_0^2 \int_0^3 (x + y) dy dx$

2- $\int_0^1 \int_0^1 (x y) dy dx$

3- $\int_0^1 \int_1^2 (x^2 + y^2) dx dy$

4- $\int_0^2 \int_0^3 (x^2 y) dx dy$