Theorem

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}.$$

Proof

$$\binom{n}{r-1}+\binom{n}{r}=\frac{n!}{(r-1)!\,(n-r+1)!}+\frac{n!}{r!\,(n-r)!}$$

$$=\frac{r\,n!}{r(r-1)!\,(n-r+1)!}+\frac{(n-r+1)n!}{r!\,(n-r)!\,(n-r+1)}$$

$$=\frac{r\,n!}{r!\,(n-r+1)!}+\frac{(n-r+1)n!}{r!\,(n-r+1)!}$$

$$=\frac{r\,n!+(n-r+1)n!}{r!\,(n-r+1)!}=\frac{n!\,(r+n-r+1)}{r!\,(n-r+1)!}$$

$$=\frac{n!\,(n+1)}{r!\,(n-r+1)!}=\frac{(n+1)!}{r!\,(n-r+1)!}$$

$$=\binom{n+1}{r}.\quad\blacksquare$$

Explanation

$$n! = n(n-1)! = n(n-1)(n-2)!$$

$$r(r-1)! = r!$$

$$(n-r+1)(n-r)! = (n-r+1)!$$

$$\binom{n+1}{r} = \frac{(n+1)!}{r!\,(n+1-r)!}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!}.$$

Example

In how many ways can a party of 7 persons arrange their selves

(i) In a row of 7 chairs.

(ii) In a circle table.

Sol.

(i) $n! = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$

(ii) $(n-1)! = (7-1)! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1.$

Theorem

Let a and $b \in R$ and $n \in Z^+$ then

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^r \,.$$

Definition 1.7

The class $\mathcal{F}$ of events (or collection, family of events) in space Ω is called "Borel field" [or sigma- field (σ- field)] in Ω if

(i) $\Omega \in \mathcal{F}$

(ii) whenever the event $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) whenever the events $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Trivial example of Borel field is $\{\phi\,,\Omega\}$.

Example

Let $S = \{H, T\}$

(i) $\mathcal{F}_1 = \{\phi\,, S\}$ is field.

(ii) $\mathcal{F}_2 = \{\phi\,, \{H\}, \{T\}, S\}$ is field.

(iii) $\mathcal{F}_3 = \{\{H\}, S\}$ is not field.

(iv) $\mathcal{F}_4 = \{\phi\,, \{H\}\}$ is not field.

Exercises

1. Prove that , if $\mathcal{F}$ is Borel field in Ω, then

(i) $\phi \in \mathcal{F}$

(ii) whenever $A_1, A_2, \ldots \in \mathcal{F}$, then also $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

(iii) whenever $A_1, A_2, \ldots, A_n \in \mathcal{F}$, then also $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ and

$\bigcap_{i=1}^{n} A_i \in \mathcal{F}$

(iv) whenever $A, B \in \mathcal{F}$, then also $A - B \in \mathcal{F}$.

2. If $A \neq \phi, \Omega$, what is the Borel field generated by $\{A\}$.

3. What is the Borel field generated by $\{\{1,2\},\{1,3\}\}$, if

(i) $\Omega = \{1,2,3,4\}$

(ii) $\Omega = \{1,2,3,4,5\}$.

Definition 1.8

The pair $(\Omega, \mathcal{F})$ of a sample space Ω together with a Borel field $\mathcal{F}$ is called a measurable space.

Definition 1.9

A probability (measure) P $\{P: \mathcal{F} \longrightarrow R\}$ is a function defined on $\mathcal{F}$ such that it assigns to each event A of $\mathcal{F}$ a positive numerical value $P(A)$.

The value $P(A)$ is a measure of the chance of the event A occurring.

Further the function P satisfies the following conditions:

(a) $0 \leq P(A) \leq 1, \forall\, A \in \mathcal{F}$.

(b) $P(\Omega) = 1$.

(c) If $A_1, A_2, \ldots \in \mathcal{F}$ are mutually exclusive (i.e.

$A_i \cap A_j = \phi, i \neq j$) then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Definition 1.10

A sample space Ω with an appropriate Borel field $\mathcal{F}$ and a given probability measures P, i.e. the triple $(\Omega, \mathcal{F}, P)$ is called a probability space and completely determines an experiment of chance.

Example

Let $S = \{HH, HT, TH, TT\}$ and $A = \{HT, TH, HH\}$ then

$$P(A) = \frac{3}{4}\,.$$

Example

A bag contains 8 red, 6 white and 7 blue balls what is the probability that two balls drawn white and blue?

Sol.

$$P = \frac{C_1^6 C_1^7}{C_2^{21}}$$

$$= \frac{\frac{6!}{1!\,5!}\ \frac{7!}{1!\,6!}}{\frac{21!}{2!\,(21-2)!}} = \frac{6 \times 7}{\frac{21!}{2!\,19!}}$$

$$= \frac{6 \times 7}{\frac{21 \times 20}{2}} = \frac{42}{210} = \frac{21}{105} = \frac{7}{35} = \frac{1}{5}.$$

Example

A coin is weighted so that head is twice as likely to appear as tail, find $P(H)$ and $P(T)$.

Sol.

$S = \{H, T\} \Longrightarrow P(S) = P(H) + P(T).$

Let $P(T) = p \Rightarrow P(H) = 2p$

$$\Rightarrow P(S) = 2p + p \Rightarrow P(S) = 3p$$

$$\because P(S) = 1 \Rightarrow 3p = 1 \Rightarrow p = \frac{1}{3} = P(T)$$

$$\Rightarrow P(H) = 2p = 2 \times \frac{1}{3} = \frac{2}{3}.$$