Theorem

(i) $P(\phi) = 0$

(ii) $P(A^c) = 1 - P(A)$, where A^c is the complement of A.

(iii) $P(A) \leq 1, \forall A \in \mathcal{F}$

(iv) Let A and $B \subseteq \Omega$, then $P(A) = P(A \cap B) + P(A \cap B^c)$

(v)For all $A, B \subseteq \Omega$, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof

(i) Any set (any event A) can be written as

$$A = A \cup \phi$$

$\Rightarrow P(A) = P(A \cup \phi)$

$\because A, \phi$ are mutually exclusive because $A \cap \phi = \phi$

$\therefore P(A) = P(A) + P(\phi)$

$\Rightarrow P(\phi) = P(A) - P(A) = 0.$

(ii) $\because S = A \cup A^c \Rightarrow P(S) = P(A \cup A^c)$

$\because A \cap A^c = \phi \Rightarrow A, A^c$ are mutually exclusive

$\therefore P(S) = P(A) + P(A^c)$

$\because P(S) = 1 \Rightarrow 1 = P(A) + P(A^c)$

$\therefore P(A^c) = 1 - P(A).$

(iii) $\because\ P(A) = 1 - P(A^c)$

$\because \{P(A^c) \geq 0\} \times (-1)$

$\Rightarrow -P(A^c) \leq 0$

$\Rightarrow 1 - P(A^c) \leq 1$

$\Rightarrow P(A) \leq 1.$

(iv) $\because A = (A \cap B) \cup (A \cap B^c)$

$\Rightarrow P(A) = P[(A \cap B) \cup (A \cap B^c)]$

$\because (A \cap B) \cap (A \cap B^c) = \phi$

$\therefore (A \cap B), (A \cap B^c)$ are mutually exclusive

$\therefore P(A) = P(A \cap B) + P(A \cap B^c)$.

(v) $A \cup B = (A \cap B^c) \cup B$

$\Longrightarrow P(A \cup B) = P[(A \cap B^c) \cup B]$

$\because (A \cap B^c)$, B are mutually exclusive

$$\therefore \; P(A \cup B) = P(A \cap B^c) + P(B) \qquad (1)$$

$$\because A = (A \cap B) \cup (A \cap B^c)$$

$\Longrightarrow P(A) = P[(A \cap B) \cup (A \cap B^c)]$

$\because (A \cap B) \cap (A \cap B^c) = \phi$

$\therefore (A \cap B), (A \cap B^c)$ are mutually exclusive

$\therefore P(A) = P(A \cap B) + P(A \cap B^c)$.

$$\Longrightarrow \; P(A \cap B^c) = P(A) - P(A \cap B) \qquad (2)$$

$\therefore\ P(A\cup B) = P(A) - P(A\cap B) + P(B)$

$\therefore\ P(A\cup B) = P(A) + P(B) - P(A\cap B).\ \blacksquare$

Example

If a die is tossed in the air and we observe the no. on the top and let:

A is the event that even no. appear.

B is the event that odd on. appear.

C is the event that prime no. appear.

Find (1) $P(A\cap B)$, (2) $P(A\cup B)$, (3) $P(A\cup C)$ and (4) $P(C^c)$.

Sol.

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\} \Rightarrow P(A) = \frac{3}{6} = \frac{1}{2}.$

$B = \{1, 3, 5\} \Rightarrow P(B) = \frac{3}{6} = \frac{1}{2}.$

$C = \{2, 3, 5\} \Rightarrow P(C) = \frac{3}{6} = \frac{1}{2}.$

(1) $A\cap B = \phi \Rightarrow A, B$are mutually exclusive

$\Rightarrow P(A \cap B) = P(\phi) = 0$.

(2) $A \cup B = \{1,2,3,4,5,6\} \Rightarrow P(A \cup B) = \frac{6}{6} = 1$.

(3) $A \cup C = \{2, 3, 4, 5, 6\} \Rightarrow P(A \cup C) = \frac{5}{6}$.

(4) $C^c = \{1, 4, 6\} \Rightarrow P(C^c) = \frac{3}{6} = \frac{1}{2}$ or

$P(C^c) = 1 - P(C) = 1 - \frac{1}{2} = \frac{1}{2}$.

Definition 1.11 (Independence)

The event A and B are independent iff

$$P(A \cap B) = P(A).P(B).$$

Example

Toss coin three times, let the event A is the first head and the event B second head and event C two heads respectively. Test (A, B) , (A, C), (B, C) are independent or no.

Sol.

$S = \{HHH, HTT, HTH, HHT, THH, THT, TTH, TTT\}$,