$\Longrightarrow P(A \cap B) = P(\phi) = 0$.

(2) $A \cup B = \{1,2,3,4,5,6\} \Longrightarrow P(A \cup B) = \frac{6}{6} = 1$.

(3) $A \cup C = \{2, 3, 4, 5, 6\} \Longrightarrow P(A \cup C) = \frac{5}{6}$.

(4) $C^c = \{1, 4, 6\} \Longrightarrow P(C^c) = \frac{3}{6} = \frac{1}{2}$ or

$P(C^c) = 1 - P(C) = 1 - \frac{1}{2} = \frac{1}{2}$.

Definition 1.11 (Independence)

The event A and B are independent iff

$$P(A \cap B) = P(A).P(B).$$

Example

Toss coin three times, let the event A is the first head and the event B second head and event C two heads respectively. Test (A, B) , (A, C), (B, C) are independent or no.

Sol.

$S = \{HHH, HTT, HTH, HHT, THH, THT, TTH, TTT\}$,

$A = \{HHH, HTT, HTH, HHT\} \Rightarrow P(A) = \frac{4}{8} = \frac{1}{2}.$

$B = \{HHH, HHT, THH, THT\} \Rightarrow P(B) = \frac{4}{8} = \frac{1}{2}.$

$C = \{HHH, HHT, THH\} \Rightarrow P(C) = \frac{3}{8}.$

$A \cap B = \{HHH, HHT\} \Rightarrow P(A \cap B) = \frac{2}{8} = \frac{1}{4}.$

$$P(A).P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = P(A \cap B).$$

$\therefore$ *A* and *B* are independent events

$A \cap C = \{HHH, HHT\} \Rightarrow P(A \cap C) = \frac{2}{8} = \frac{1}{4}.$

$$P(A)P(C) = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) = \frac{3}{16} \neq P(A \cap C).$$

$\therefore$ *A* and *C* are not independent events.

$B \cap C = \{HHH, HHT, THH\} \Rightarrow P(B \cap C) = \frac{3}{8}.$

$$P(B)P(C) = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) = \frac{3}{16} \neq P(B \cap C).$$

$\therefore$ *B* and *C* are not independent events.

Theorem

If A and B are independent events, then A and B^c are independent.

Proof

$\because P(A) = P(A \cap B) + P(A \cap B^c)$

$\Rightarrow P(A \cap B^c) = P(A) - P(A \cap B)$

$\because$ A and B are independent events

$\therefore P(A \cap B) = P(A).P(B)$

$\therefore P(A \cap B^c) = P(A) - P(A).P(B)$

$= P(A)\big(1 - P(B)\big)$

$= P(A).P(B^c).$ ■

$\therefore$ A and B^c are independent events.

Definition 1.12

The events $A_1, A_2, \dots, A_n$ are said to be pairwise independent if for any i and j , $1 \leq i, j \leq n$, $i \neq j$, $P(A_i \cap A_j) = P(A_i).P(A_j)$.

Note that pairwise independent does not imply mutually independent.

Example

Let $\Omega = \{a, b, c, d\}$. Let

$$P(A) = \frac{the\ number\ of\ point\ in\ A}{4}, \forall\ A \subset \Omega.$$

If $A = \{a, d\}$, $B = \{b, d\}$ and $C = \{c, d\}$.

$\Rightarrow P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{2}{4} = \frac{1}{2}$ and $P(C) = \frac{2}{4} = \frac{1}{2}$.

$$\therefore P(A) = P(B) = P(C) = \frac{1}{2}.$$

$P(A \cap B) = \frac{1}{4}$, $P(A \cap C) = \frac{1}{4}$ and $P(B \cap C) = \frac{1}{4}$.

$$\therefore P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4},$$

then A, B , C are pairwise independent.

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A).P(B).P(C) = \frac{1}{8}.$$

Theorem

The two events A and B are independent iff A^c and B^c are independent.

Proof

$\Longrightarrow$

Let A and B are independent and we prove that A^c and B^c are independent.

$$\because P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)].$$

$\because$ A and B are independent $\Longrightarrow P(A \cap B) = P(A). P(B)$.

$$\therefore P(A^c \cap B^c) = 1 - [P(A) + P(B) - P(A). P(B)]$$

$$= 1 - P(A) - P(B) + P(A). P(B)$$

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= [1 - P(A)][1 - P(B)]$$

$$= P(A^c). P(B^c).$$

$\therefore$ A^c and B^c are independent.

$\Longleftarrow$

Let A^c and B^c are independent we prove that A and B are independent.

$\because P(A \cap B) = 1 - P(A \cap B)^c$

$$= 1 - P(A^c \cup B^c)$$

$$= 1 - [P(A^c) + P(B^c) - P(A^c \cap B^c)]$$

$\because A^c$ and B^c are independent $\Longrightarrow P(A^c \cap B^c) = P(A^c).P(B^c)$

$$\therefore\ P(A \cap B) = 1 - [P(A^c + P(B^c) - P(A^c).P(B^c)]$$

$$\Longrightarrow P(A \cap B) = 1 - P(A^c) - P(B^c) + P(A^c).P(B^c)$$

$$= [1 - P(A^c)] - P(B^c)[1 - P(A^c)]$$

$$= [1 - P(A^c)][1 - P(B^c)]$$

$$= P(A).P(B).$$

$\therefore A$ and B are independent. ■