The conditional probability

Definition 1.13

The conditional probability of A given B written as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0.$$

Example

Two dice are tossed and let the event A that the number 2 appear at least in one die and the event B that get sum 6. Find

(1) $P(A|B)$,

(2) $P(B|A)$.

Sol.

$A = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)\}$

$\Longrightarrow P(A) = \frac{11}{36}$.

$B = \{(2,4), (4,2), (1,5), (5,1), (3,3)\} \Longrightarrow P(B) = \frac{5}{36}$.

$A \cap B = \{(2,4), (4,2)\} \Longrightarrow P(A \cap B) = \frac{2}{36}$.

$$(1)\ P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}.$$

$$(2)\ P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

Theorem

If A and B are independent events, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

Proof

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

$\because$ A and B are independent events

$$\therefore P(A \cap B) = P(A).P(B).$$

$$\therefore P(A|B) = \frac{P(A).P(B)}{P(B)} = P(A).$$

And

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A).P(B)}{P(A)} = P(B). \quad \blacksquare$$

Theorem

Let A and B two events such that $P(B) > 0$, then

(1) $P(B|S) = P(B)$.

(2) $P(A^c|B) = 1 - P(A|B)$.

(3) $P(S|B) = 1$.

(4) $P(\phi|B) = 0$.

(5) $P(S|S) = 1$.

Proof

(1) $P(B|S) = \dfrac{P(B \cap S)}{P(S)} = \dfrac{P(B)}{P(S)}$

$\because P(S) = 1 \Longrightarrow P(B|S) = \dfrac{P(B)}{1} = P(B).$

(2) $P(A^c|B) = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)}$

$$= \frac{P(B)}{P(B)} - \frac{P(A \cap B)}{P(B)} = 1 - P(A|B).$$

(3) $P(S|B) = \dfrac{P(B \cap S)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

(4) $P(\phi|B) = \dfrac{P(\phi \cap B)}{P(B)} = \dfrac{P(\phi)}{P(B)} = \dfrac{0}{P(B)} = 0\,.$

(5) $P(S|S) = \dfrac{P(S \cap S)}{P(S)} = \dfrac{P(S)}{P(S)} = 1\,.$

Theorem

Let A , B and C three events such that $P(C) \neq 0$, then

$$P[(A \cup B)|C] = P(A|C) + P(B|C) - P[(A \cap B)|C].$$

Proof

$$P[(A \cup B)|C] = \frac{P[(A \cup B) \cap C]}{P(C)}$$

$$= \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)}$$

$$= \frac{P(A \cap C) + P(B \cap C) - P[(A \cap B) \cap C]}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P[(A \cap B) \cap C]}{P(C)}$$

$$= P(A|C) + P(B|C) - P[(A \cap B)|C]. \quad \blacksquare$$

Theorem

If $P(A) = a$ and $P(B) = b$, then

$$P(A|B) \geq \frac{a+b-1}{b}.$$

Proof

$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\because P(A \cup B) \leq 1 \Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow \ -P(A \cap B) \leq 1 - P(A) - P(B)\} \times (-1)$

$\Rightarrow P(A \cap B) \geq \ P(A) + P(B) - 1\} \div P(B)$

$\Rightarrow \dfrac{P(A \cap B)}{P(B)} \geq \dfrac{P(A) + P(B) - 1}{P(B)}$

$$\Rightarrow P(A|B) \geq \frac{a+b-1}{b}. \quad \blacksquare$$

Decomposition Theorem

If $\{A_n\}$ is a sequence of mutually exclusive (disjoint) events with $\bigcup_{i=1}^{n} A_i = \Omega$ and $P(A_i) \geq 0$, $i = 1,2, \dots, n$,then for any event $B \in \mathcal{F}$,

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i).$$

Proof

$$B = B \cap \Omega$$

$$= B \cap [\bigcup_{i=1}^{n} A_i]$$

$$= \bigcup_{i=1}^{n} (B \cap A_i).$$

Since $\{A_n\}$ are disjoint

$$\therefore P(B) = P[\bigcup_{i=1}^{n} (B \cap A_i)]$$

$$= \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(A_i) P(B|A_i). \quad \blacksquare$$

Bayes' Theorem

If $\{A_n\}$ is a sequence of disjoint events with $\bigcup_{i=1}^{n} A_i = \Omega$ and $P(A_i) > 0$, $i = 1,2, \ldots, n$, then for any event $B \in \mathcal{F}$

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}, i = 1,2, \ldots, n\,.$$

Proof

Since

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)},$$

$\because P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)}$

$\Longrightarrow P(B \cap A_i) = P(A_i).P(B|A_i)$

$\because P(B \cap A_i) = P(A_i \cap B)$

$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$

$\because P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$ by decomposition theorem

$$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}\,. \quad \blacksquare$$