



Electromagnetic Theory

4th year

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ملاحظات حول الامتحانات الشهرية:

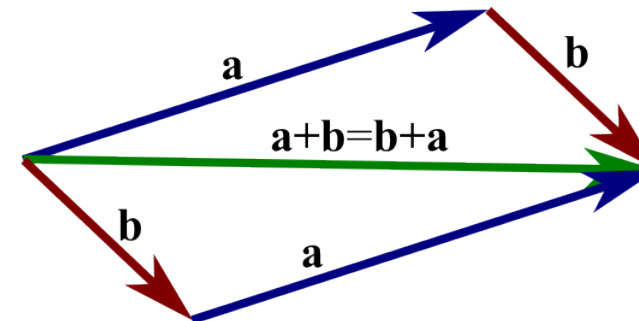
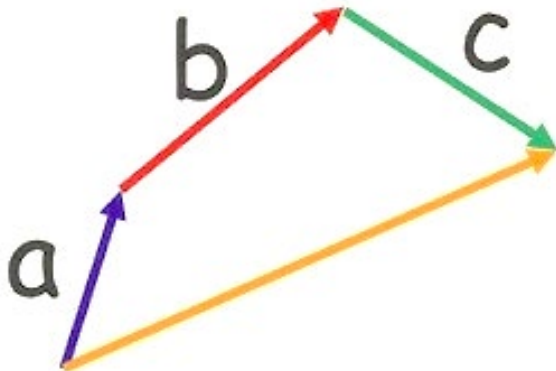
- 1- يمنع استعارة القرطاسية (القلم والممحاة والمبراة)
 - 2- يمنع استعارة الحاسبة منعاً باتاً
 - 3- يمنع استخدام ورقة إضافية غير الورقة الامتحانية
 - 4- يغلق الهاتف او يوضع على وضعية الطيران ويتم وضعة خارج المقاعد الدراسية عند الاستيغ
 - 5- نذكركم بالتعليمات الجامعية الاتية :
- المادة (9) : يعتبر الطالب راسباً في أي موضوع إذا تجاوزت غياباته (10%) عشر من المئة من الساعات المقررة لذلك الموضوع بدون عذر مشروع أو (15%) خمس عشره من المئة بعذر مشروع يقره مجلس الكلية أو المعهد
- المادة (20) : إذا ثبت غش الطالب أو ثبتت محاولته الغش في أي من الامتحانات اليومية أو الأسبوعية أو الشهرية أو الفصلية أو النهائية يعتبر راسباً في جميع المواضيع لتلك السنة وإذا تكرر ذلك يفصل من الكلية أو المعهد ويرقن قيده من سجلاتها .

1-Quantities

A Scalar quantity is a quantity which is have only magnitude.

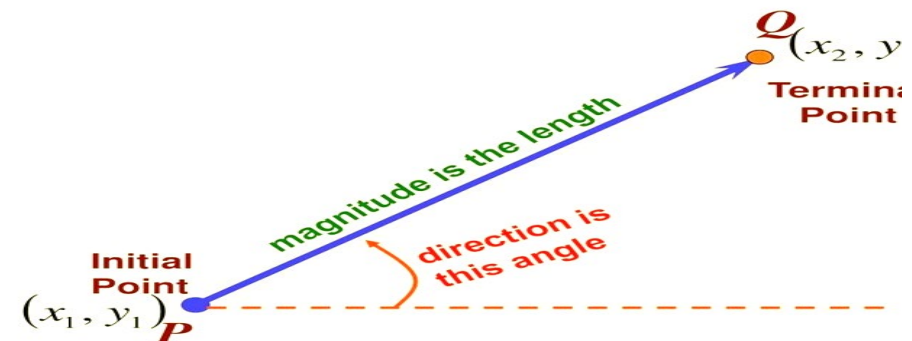
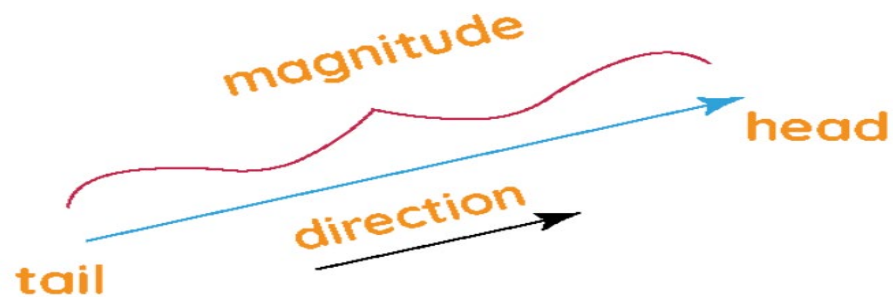
A Vector quantity is a quantity which is have magnitude and direction.

Laws of Vector Algebra	
Commutative law for addition	$\vec{A} + \vec{B} = \vec{B} + \vec{A}$
Associative law for addition	$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$
Associative law for addition	$m(n\vec{A}) = n(m\vec{A})$
Distributive law	$(m + n)\vec{A} = m\vec{A} + n\vec{A}$ or $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$
Distributive law	$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$



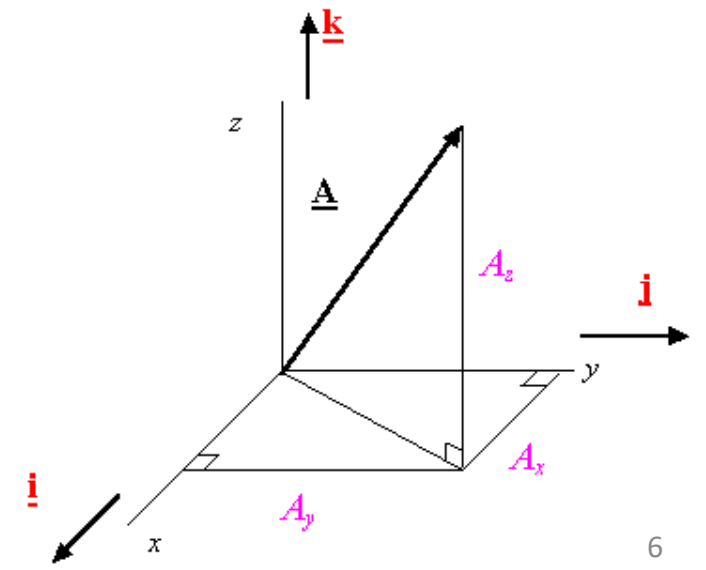
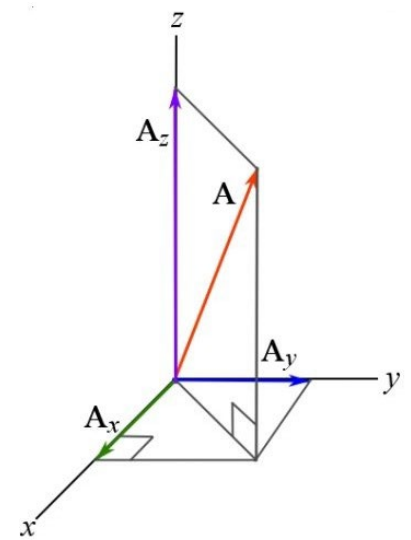
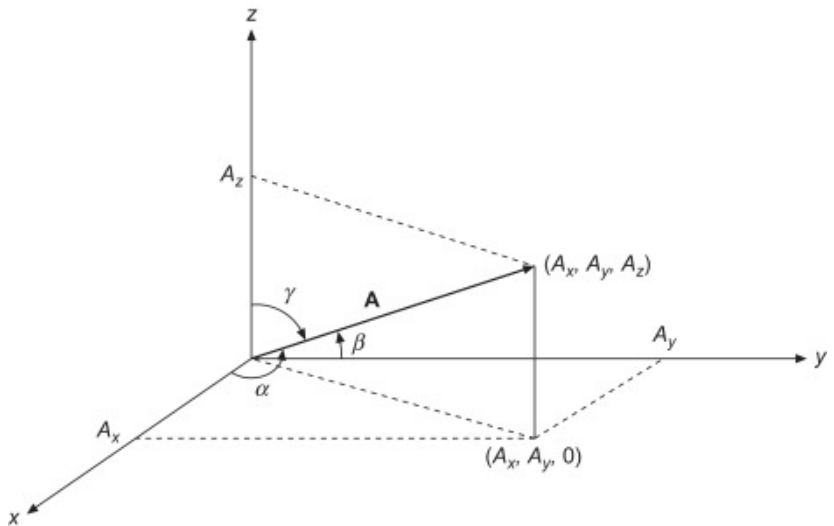
2-Vectors Definitions

	Vector	a quantity that has both magnitude (size) and direction. It's often represented graphically as an arrow, where the arrow's length corresponds to the magnitude and the arrowhead indicates the direction.
	Vectors algebra	Involves algebraic operations across vectors
	Magnitude of the vector	The length of the line between the two points P and Q
	Direction of the vector	The direction or the <u>displacement</u> of point P to point Q
	The tail	The initial point of a vector P
	The head	The terminal point Q
	Initial point	Start position of the vector P
	Terminal point	Final position of the vector Q



3-Vector Algebra

	Representation of Vectors Or Components of Vector	$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$
	Direction of Unit Vectors	$\hat{i}, \hat{j}, \hat{k}$
	Unit Vectors \vec{u}	Vectors that have magnitude equals to 1
	The magnitude of \vec{A}	$A = \vec{A} = \sqrt{A_x^2 + A_y^2 + A_z^2}$
	The position vector or radius vector \vec{r} from 0	$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
	The direction of a vector	The cosine values of $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines of the vector \vec{A}



Example 1:

Given a vector $A = 3\hat{i} + 4\hat{j} + 5\hat{k}$, find the following:

1. The **magnitude** of vector A .
2. The **unit vector** in the direction of A .
3. The **direction cosines** of vector A .

Solution Example 1:

1. Magnitude of Vector A

The magnitude of a vector $A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ is given by the formula $A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

For our vector $A = 3\hat{i} + 4\hat{j} + 5\hat{k}$, the components are $A_x = 3$, $A_y = 4$, and $A_z = 5$.

$$A = |A| = \sqrt{(3)^2 + (4)^2 + (5)^2}$$

$$A = \sqrt{9 + 16 + 25}$$

$$A = \sqrt{50}$$

$$A = 5\sqrt{2} \approx 7.07$$

2. Unit Vector in the Direction of A

A **unit vector** \hat{u} in the direction of any vector A is defined as the vector divided by its magnitude:

$$\hat{u} = \frac{A}{|A|}.$$

Using our vector $A = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and its magnitude $|A| = \sqrt{50}$, the unit vector \hat{A} is:

$$\hat{A} = \frac{3\hat{i} + 4\hat{j} + 5\hat{k}}{\sqrt{50}}$$

$$\hat{A} = \frac{3}{\sqrt{50}}\hat{i} + \frac{4}{\sqrt{50}}\hat{j} + \frac{5}{\sqrt{50}}\hat{k}$$

3. Direction Cosines of Vector A

The direction cosines of a vector $V = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ are given by the formulas:

- $\cos(\alpha) = \frac{V_x}{|V|}$
- $\cos(\beta) = \frac{V_y}{|V|}$
- $\cos(\gamma) = \frac{V_z}{|V|}$

Where α , β , and γ are the angles the vector makes with the positive x , y , and z axes, respectively.

Using our vector $A = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and its magnitude $|A| = \sqrt{50}$:

- **Direction cosine with the x -axis ($\cos(\alpha)$):**

$$\cos(\alpha) = \frac{3}{\sqrt{50}}$$

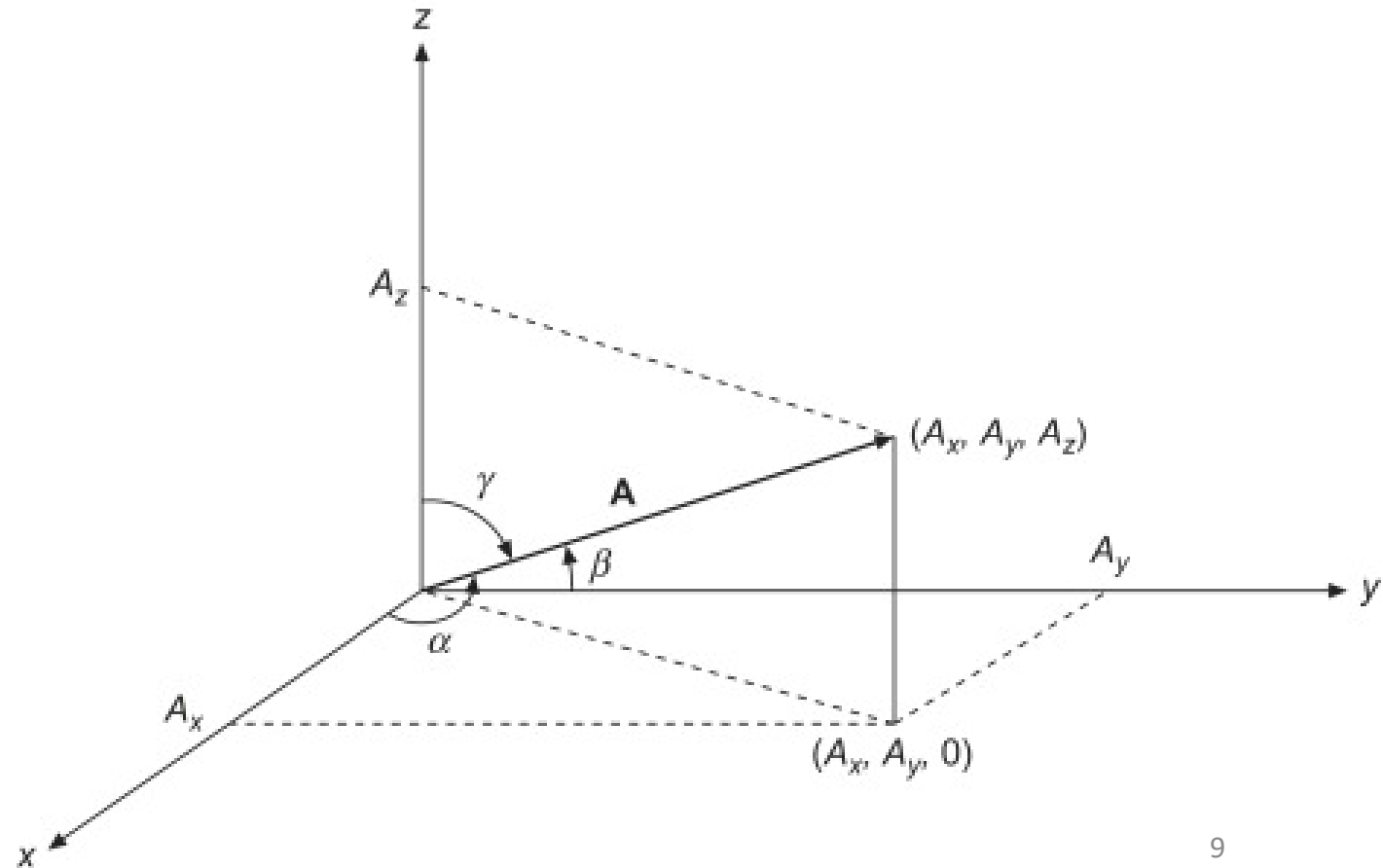
- **Direction cosine with the y -axis ($\cos(\beta)$):**

$$\cos(\beta) = \frac{4}{\sqrt{50}}$$

- **Direction cosine with the z -axis ($\cos(\gamma)$):**

$$\cos(\gamma) = \frac{5}{\sqrt{50}}$$

So, the direction cosines of vector A are $\frac{3}{\sqrt{50}}$, $\frac{4}{\sqrt{50}}$, and $\frac{5}{\sqrt{50}}$.



Example 2:

Prove that the vector directed from $M(x_2, y_2, z_2)$ to $N(x_1, y_1, z_1)$ is given by; $(x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$.

Solution:

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

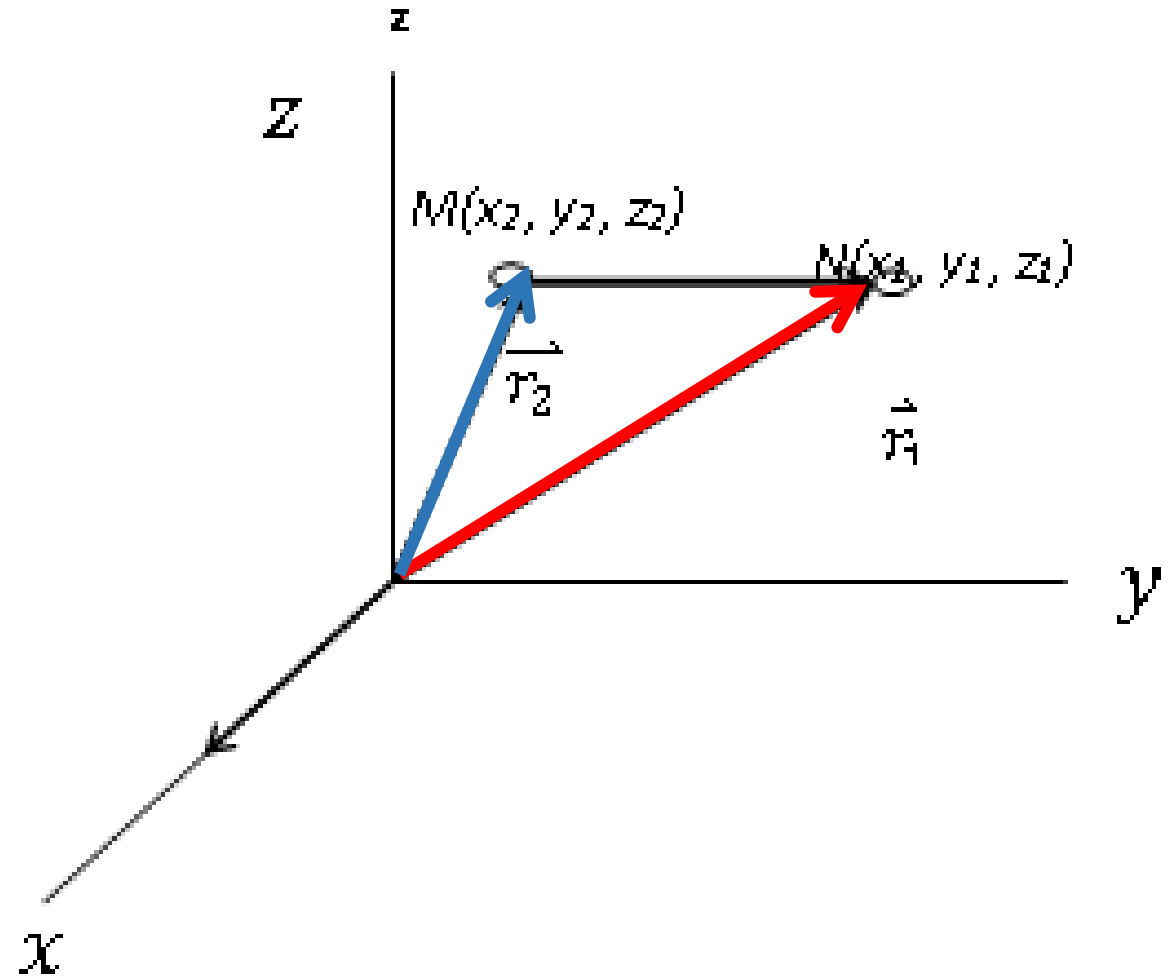
$$\vec{r}_1 = \vec{r}_2 + \vec{r}_{21}$$

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$$

$$\vec{r}_{21} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$$

$$\vec{r}_{21} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} - x_2\hat{i} - y_2\hat{j} - z_2\hat{k}$$

$$\vec{r}_{21} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

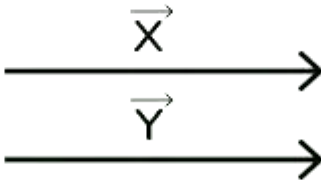


4-Types of Vectors

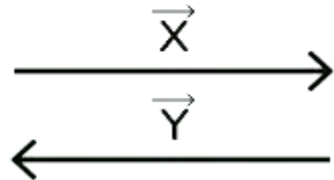
Zero Vectors	Vectors that have Zero magnitude
Parallel Vectors	Vectors that have the same or opposite direction.
Equal Vectors $\vec{A} = \vec{B}$	Vectors have the same magnitude and same direction
Negative Vectors $\vec{A} = -\vec{B}$	Vectors have the same magnitudes but opposite directions
Orthogonal (Perpendicular) Vectors	Vectors that are at right angles (90 degrees) to each other.
Collinear Vectors	Vectors that lie along the same line or are parallel to each other.
Coplanar Vectors	Vectors that lie in the same plane.
Co-initial Vectors	Vectors that have the same initial point.



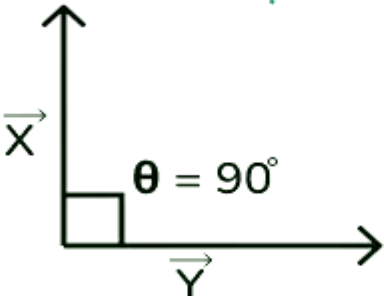
 Zero Vector




 Equal Vectors



 Negative Vectors



 Orthogonal Vector



 Collinear Vector

Example 3:

Given two vectors $A = 2\hat{i} + 3\hat{j}$ and $B = 4\hat{i} - \hat{j}$, and two scalars $m = 2$ and $n = 3$, verify the following laws:

1. **Commutative Law for Addition:** $A + B = B + A$
2. **Distributive Law:** $m(A + B) = mA + mB$
3. **Associative Law for Addition:** $(mA + nA) = (m + n)A$

1. Commutative Law for Addition

This law states that the order in which vectors are added does not affect the result.

- **Left-Hand Side (LHS):** $A + B$

$$A + B = (2\hat{i} + 3\hat{j}) + (4\hat{i} - \hat{j})$$

$$A + B = (2 + 4)\hat{i} + (3 - 1)\hat{j}$$

$$A + B = 6\hat{i} + 2\hat{j}$$

- **Right-Hand Side (RHS):** $B + A$

$$B + A = (4\hat{i} - \hat{j}) + (2\hat{i} + 3\hat{j})$$

$$B + A = (4 + 2)\hat{i} + (-1 + 3)\hat{j}$$

$$B + A = 6\hat{i} + 2\hat{j}$$

Since LHS = RHS ($6\hat{i} + 2\hat{j} = 6\hat{i} + 2\hat{j}$), the **Commutative Law for Addition** is verified.

2. Distributive Law

This law shows how a scalar multiplication distributes over vector addition.

- **Left-Hand Side (LHS):** $m(A + B)$

First, calculate the sum of the vectors: $A + B = 6\hat{i} + 2\hat{j}$ (from the previous step).

Now, multiply by the scalar $m = 2$:

$$m(A + B) = 2(6\hat{i} + 2\hat{j})$$

$$m(A + B) = (2 \cdot 6)\hat{i} + (2 \cdot 2)\hat{j}$$

$$m(A + B) = 12\hat{i} + 4\hat{j}$$

- **Right-Hand Side (RHS):** $mA + mB$

First, perform the scalar multiplications:

$$mA = 2(2\hat{i} + 3\hat{j}) = 4\hat{i} + 6\hat{j}$$

$$mB = 2(4\hat{i} - \hat{j}) = 8\hat{i} - 2\hat{j}$$

Now, add the resulting vectors:

$$mA + mB = (4\hat{i} + 6\hat{j}) + (8\hat{i} - 2\hat{j})$$

$$mA + mB = (4 + 8)\hat{i} + (6 - 2)\hat{j}$$

$$mA + mB = 12\hat{i} + 4\hat{j}$$

Since LHS = RHS ($12\hat{i} + 4\hat{j} = 12\hat{i} + 4\hat{j}$), the **Distributive Law** is verified.

3. Associative Law for Addition

$m(nA) = (mn)A$. We'll verify this using the form $(mA + nA) = (m + n)A$

- **Left-Hand Side (LHS):** $mA + nA$

$$mA = 2(2\hat{i} + 3\hat{j}) = 4\hat{i} + 6\hat{j}$$

$$nA = 3(2\hat{i} + 3\hat{j}) = 6\hat{i} + 9\hat{j}$$

$$mA + nA = (4\hat{i} + 6\hat{j}) + (6\hat{i} + 9\hat{j})$$

$$mA + nA = (4 + 6)\hat{i} + (6 + 9)\hat{j}$$

$$mA + nA = 10\hat{i} + 15\hat{j}$$

- **Right-Hand Side (RHS):** $(m + n)A$

First, add the scalars: $m + n = 2 + 3 = 5$.

Now, multiply the vector by this sum:

$$(m + n)A = 5(2\hat{i} + 3\hat{j})$$

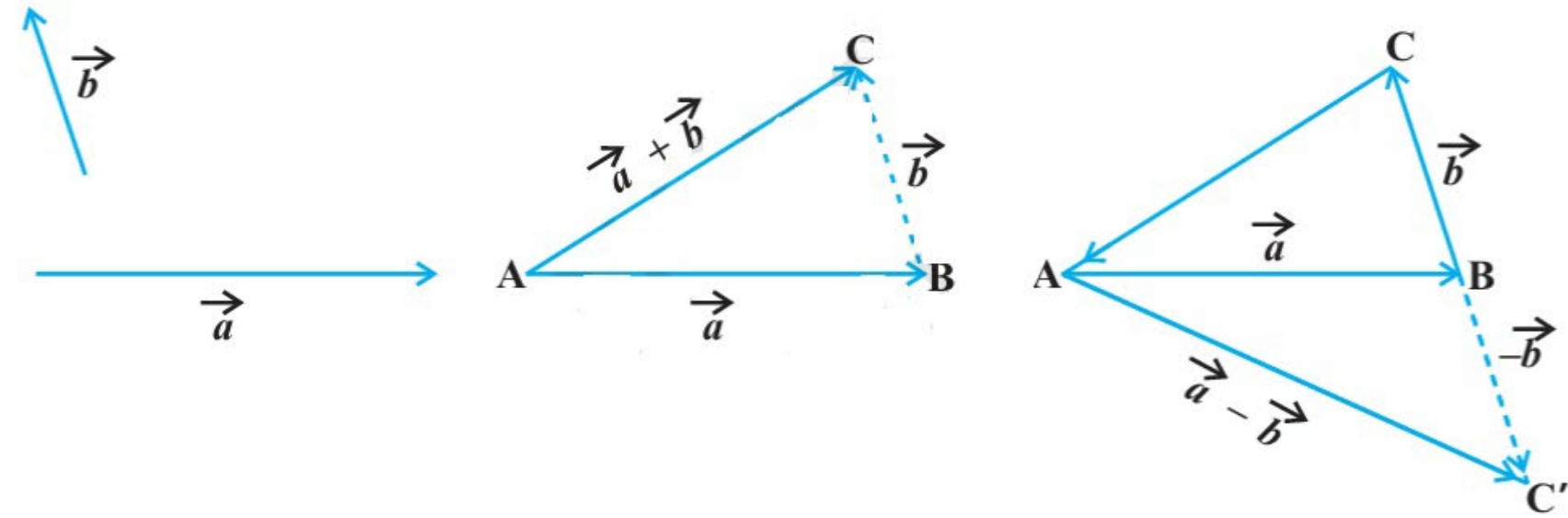
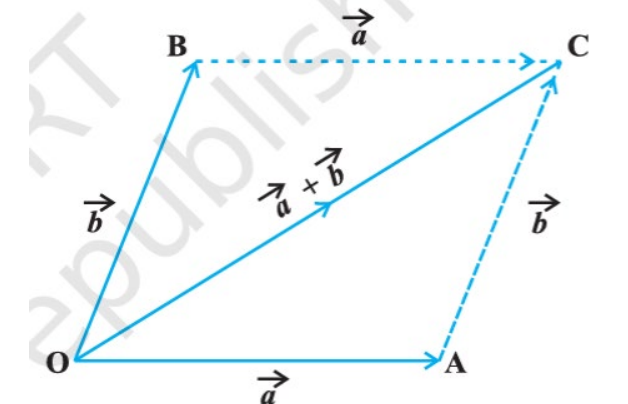
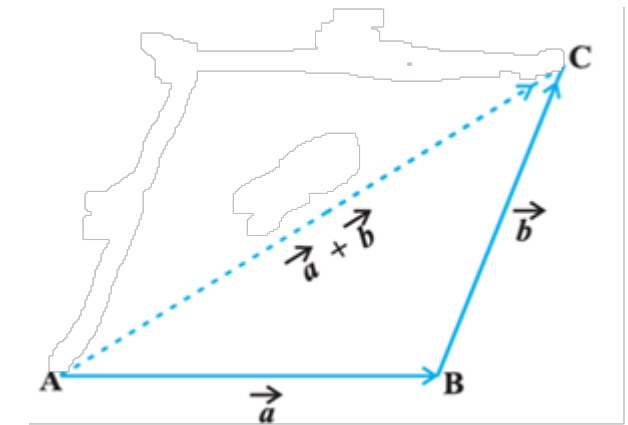
$$(m + n)A = (5 \cdot 2)\hat{i} + (5 \cdot 3)\hat{j}$$

$$(m + n)A = 10\hat{i} + 15\hat{j}$$

Since LHS = RHS ($10\hat{i} + 15\hat{j} = 10\hat{i} + 15\hat{j}$), this variant of the **Associative Law for Addition** is verified.

5-Vector Algebra Operations

Vector Addition	Vector Subtraction
$\vec{C} = \vec{A} + \vec{B}$	$\vec{D} = \vec{A} - \vec{B}$
$\vec{A} + \vec{B}$ $= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$ $+ (A_z + B_z)\hat{k}$	$\vec{A} - \vec{B}$ $= (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}$ $+ (A_z - B_z)\hat{k}$
Triangle Law of Addition of Vectors	$\vec{AC} = \vec{a} + \vec{b}$
Parallelogram Law of Addition of Vectors	$\vec{OC} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$



Example 4:

Given two vectors $A = 3\hat{i} + 2\hat{j}$ and $B = \hat{i} + 4\hat{j}$, find the following:

1. The vector sum $C = A + B$.
2. The vector difference $D = A - B$.
3. Describe how the **Triangle Law** and **Parallelogram Law** of addition apply to these vectors.

Solution

1. Vector Addition

To find the vector sum $C = A + B$, we use the component-wise addition formula from the table:

$$C = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

Given $A = 3\hat{i} + 2\hat{j}$ and $B = \hat{i} + 4\hat{j}$:

$$C = (3 + 1)\hat{i} + (2 + 4)\hat{j}$$

$$C = 4\hat{i} + 6\hat{j}$$

The resultant vector is $4\hat{i} + 6\hat{j}$.

2. Vector Subtraction

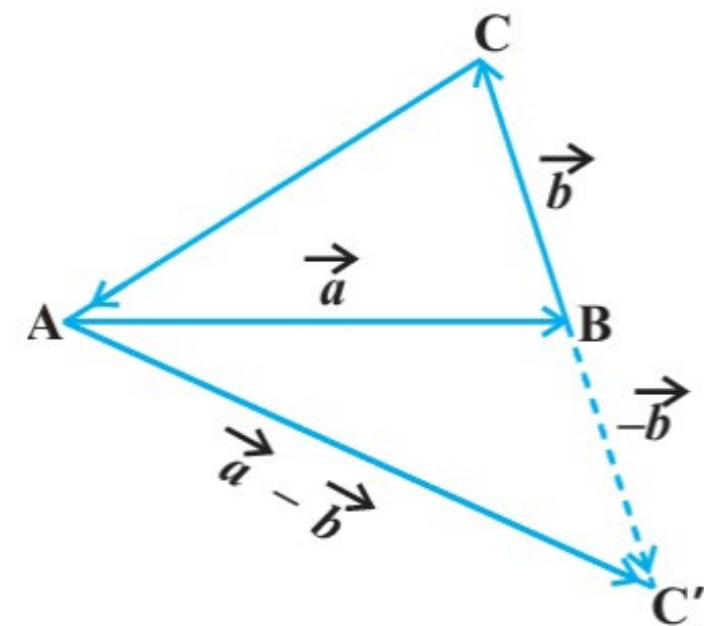
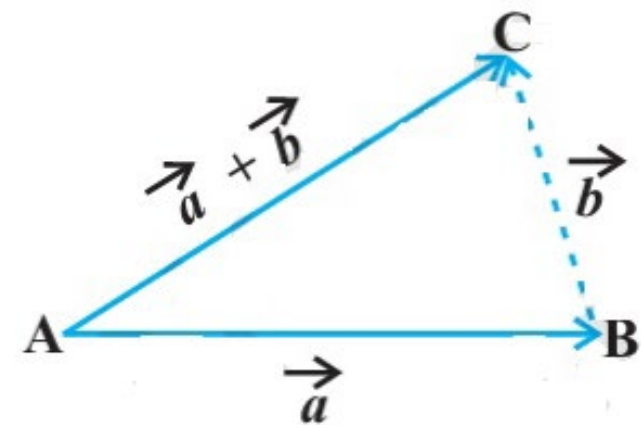
To find the vector difference $D = A - B$, we use the component-wise subtraction formula:

$$D = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}$$

$$D = (3 - 1)\hat{i} + (2 - 4)\hat{j}$$

$$D = 2\hat{i} - 2\hat{j}$$

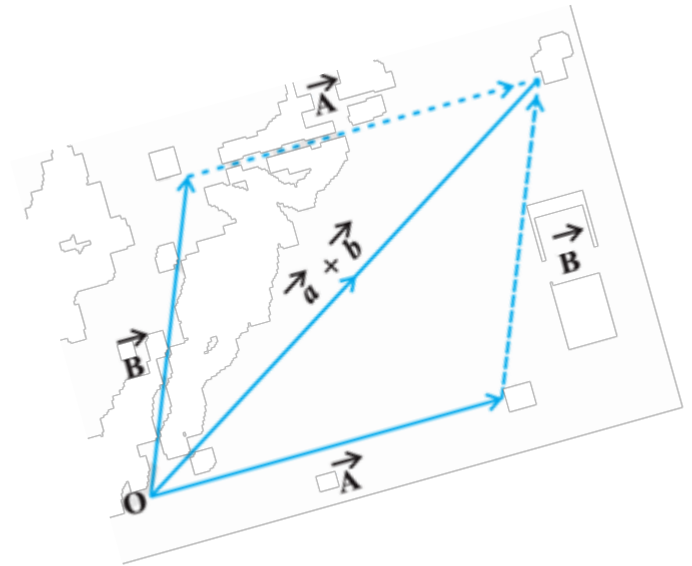
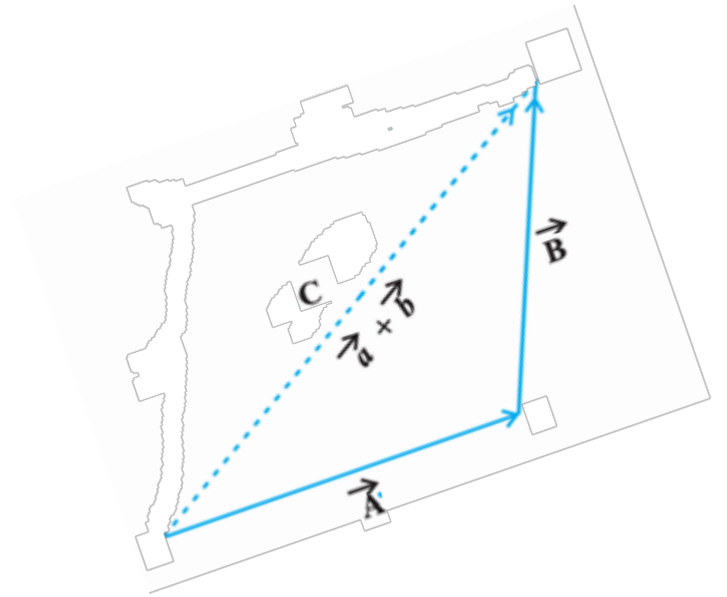
The resultant vector is $2\hat{i} - 2\hat{j}$.



3. Geometric Laws of Addition

Both the Triangle Law and Parallelogram Law provide a visual representation of vector addition.

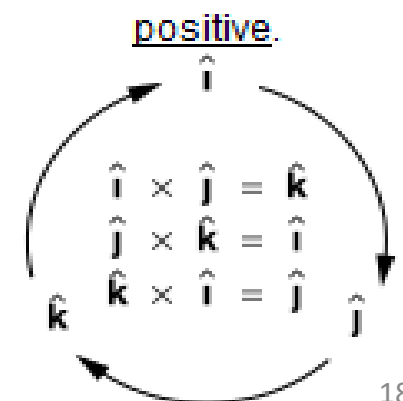
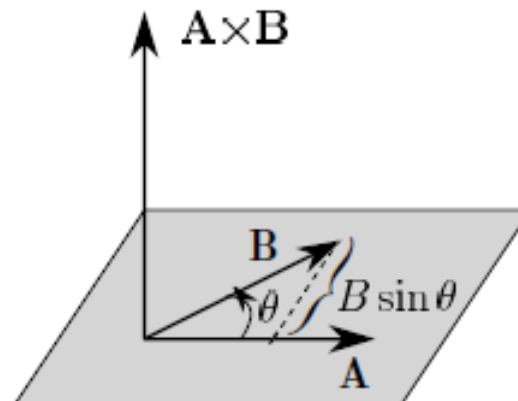
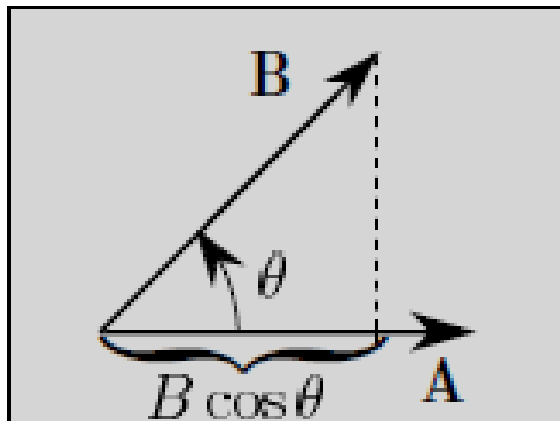
- **Triangle Law of Addition:** This law states that if we place the tail of vector B at the head of vector A , the resultant vector $C = A + B$ is the vector that goes from the tail of A to the head of B .
 - To apply this, imagine vector A starts at the origin $(0, 0)$ and ends at $(3, 2)$.
 - Vector B then starts at $(3, 2)$ and ends at $(3 + 1, 2 + 4)$, which is $(4, 6)$.
 - The resultant vector C is a straight line from the origin $(0, 0)$ to the final point $(4, 6)$, which corresponds to the vector $4\hat{i} + 6\hat{j}$ we calculated earlier.
- **Parallelogram Law of Addition:** This law states that if two vectors start from the same origin, their sum is the diagonal of the parallelogram they form.
 - Vectors A and B both start from the origin $(0, 0)$, ending at $(3, 2)$ and $(1, 4)$ respectively.
 - To form the parallelogram, we draw a vector parallel to A from the head of B (at $(1, 4)$) and a vector parallel to B from the head of A (at $(3, 2)$). These two lines meet at the point $(4, 6)$.
 - The diagonal of this parallelogram, starting from the origin and ending at $(4, 6)$, is the resultant vector $C = 4\hat{i} + 6\hat{j}$. The other diagonal of the parallelogram, from the head of B to the head of A , is the difference vector $D = A - B = 2\hat{i} - 2\hat{j}$.



6-Vector Algebra Operations

6- Multiplication of Vectors

Dot (Scalar) Product	Cross (Vector) Product
$\vec{A} \cdot \vec{B} = \vec{A} \vec{B} \cos\theta$	$\vec{A} \times \vec{B} = \vec{A} \vec{B} \sin\theta \hat{u}$
$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$	$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$	$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_x B_z - A_z B_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$
$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$	$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
$m(\vec{A} \cdot \vec{B}) = (m\vec{A}) \cdot \vec{B} = \vec{A} \cdot (m\vec{B}) = (\vec{A} \cdot \vec{B})m$	$m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) = (\vec{A} \times \vec{B})m$
$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$	$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$



Example 5:

Given two vectors $A = 2\hat{i} + 3\hat{j} + \hat{k}$ and $B = 4\hat{i} - \hat{j} + 2\hat{k}$, find the following:

1. The **Dot (Scalar) Product** $A \cdot B$.
2. The angle θ between vectors A and B .
3. The **Cross (Vector) Product** $A \times B$.
4. Verify the **Anticommutative Property** of the cross product by calculating $B \times A$.
5. Verify the **Distributive Law** by calculating $A \cdot (A + B)$ in two different ways.

1. Dot Product: $A \cdot B$

We use the component-wise formula from the table: $A \cdot B = A_x B_x + A_y B_y + A_z B_z$.

$$A \cdot B = (2)(4) + (3)(-1) + (1)(2) = 8 - 3 + 2 = 7$$

2. Angle between A and B

The dot product can also be expressed as $A \cdot B = |A||B| \cos \theta$. We can rearrange this to find

the angle: $\cos \theta = \frac{A \cdot B}{|A||B|}$.

First, find the magnitudes of the vectors:

- $|A| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$
- $|B| = \sqrt{4^2 + (-1)^2 + 2^2} = \sqrt{16 + 1 + 4} = \sqrt{21}$

Now, substitute the values into the formula:

$$\cos \theta = \frac{7}{\sqrt{14}\sqrt{21}} = \frac{7}{\sqrt{294}} = \frac{7}{7\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{6}}\right) \approx 65.9^\circ$$

3. Cross Product: $A \times B$

We use the determinant formula from the table:

$$\begin{aligned}A \times B &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 4 & -1 & 2 \end{vmatrix} \\ &= \hat{i}((3)(2) - (1)(-1)) - \hat{j}((2)(2) - (1)(4)) + \hat{k}((2)(-1) - (3)(4)) \\ &= \hat{i}(6 - (-1)) - \hat{j}(4 - 4) + \hat{k}(-2 - 12) \\ &= 7\hat{i} + 0\hat{j} - 14\hat{k} = 7\hat{i} - 14\hat{k}\end{aligned}$$

4. Anticommutative Property: $B \times A$

According to the table, $B \times A = -A \times B$. Let's verify this by calculating $B \times A$.

$$\begin{aligned}B \times A &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ 2 & 3 & 1 \end{vmatrix} \\ &= \hat{i}((-1)(1) - (2)(3)) - \hat{j}((4)(1) - (2)(2)) + \hat{k}((4)(3) - (-1)(2)) \\ &= \hat{i}(-1 - 6) - \hat{j}(4 - 4) + \hat{k}(12 - (-2)) \\ &= -7\hat{i} - 0\hat{j} + 14\hat{k} = -7\hat{i} + 14\hat{k}\end{aligned}$$

Since $- (7\hat{i} - 14\hat{k}) = -7\hat{i} + 14\hat{k}$, the property is verified.

5. Distributive Law: $A \cdot (A + B)$

Method 1: Direct Calculation

First, find the sum of the vectors:

$$A + B = (2\hat{i} + 3\hat{j} + \hat{k}) + (4\hat{i} - \hat{j} + 2\hat{k}) = 6\hat{i} + 2\hat{j} + 3\hat{k}$$

Now, take the dot product with A :

$$\begin{aligned} A \cdot (A + B) &= (2\hat{i} + 3\hat{j} + \hat{k}) \cdot (6\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= (2)(6) + (3)(2) + (1)(3) = 12 + 6 + 3 = 21 \end{aligned}$$

Method 2: Using the Distributive Law

According to the table, $A \cdot (A + B) = A \cdot A + A \cdot B$.

- **Calculate $A \cdot A$:**

$$A \cdot A = |A|^2 = (\sqrt{14})^2 = 14$$

- **Use the result from Part 1:**

$$A \cdot B = 7$$

- **Add the two results:**

$$A \cdot A + A \cdot B = 14 + 7 = 21$$

Both methods yield the same result, thus the distributive law is verified.

Example 6:

Prove that $|\vec{A} \times \vec{B}|$ represent the area of a parallelogram with sides \vec{A} and \vec{B} .

Solution:

Area of parallelogram=Height x Base. i.e.

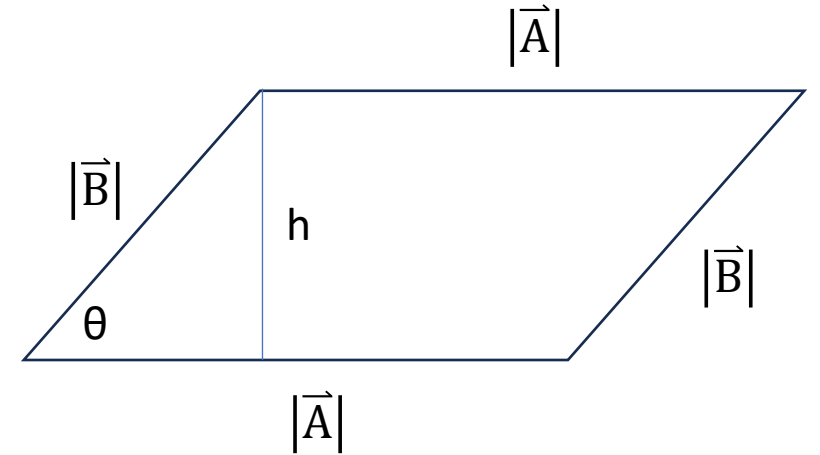
$$\text{Area} = h \cdot |\vec{A}|$$

$$h = |\vec{B}| \sin \theta$$

$$\text{Area} = |\vec{B}| \sin \theta \cdot |\vec{A}|$$

$$\text{Area} = |\vec{B}| |\vec{A}| \sin \theta$$

$$\text{Area} = |\vec{A} \times \vec{B}|$$



7-Vector Algebra Operations (Triple Product)

Dot (Scalar) Product	Cross (Vector) Product
$(\vec{A} \cdot \vec{B})\vec{C} \neq \vec{C}(\vec{B} \cdot \vec{A})$	$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$
$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) =$	$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
H.W. Prove all (5)	$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$



Example 7:

Prove that $\vec{A} \cdot (\vec{B} \times \vec{C})$ is in absolute value equal to the volume of parallelepiped with sides \vec{A} , \vec{B} and \vec{C} .

Solution:

- The volume of a parallelepiped is given by: **Volume = (Area of the Base) \times (Height)**
- Let's choose the **base** of our parallelepiped to be the parallelogram formed by vectors \vec{B} and \vec{C} .

Therefore, the **Area of the Base** is $|\vec{B} \times \vec{C}|$.

- The **height** (h) is the perpendicular distance from the tip of \vec{A} down to the base plane (the plane containing \vec{B} and \vec{C})
- The height is the component of \vec{A} that is perpendicular to the base.

The vector \hat{n} is perpendicular to the base.

The component of \vec{A} in the direction of \hat{n} is given by the dot product: $\vec{A} \cdot \hat{n} = |\vec{A}| \cos \theta$, where θ is the angle between \vec{A} and \hat{n} .

This component, $|\vec{A}| \cos \theta$, is exactly the **height** (h) of the parallelepiped.

(If \vec{A} lies almost flat in the base plane, θ is large, $\cos \theta$ is small, and the volume is small. If \vec{A} is perfectly perpendicular to the base, $\theta = 0$, $\cos \theta = 1$, and the height is maximum.)

So,

$$\text{Height}(h) = |\vec{A}| \cos \theta = \vec{A} \cdot \hat{n}$$

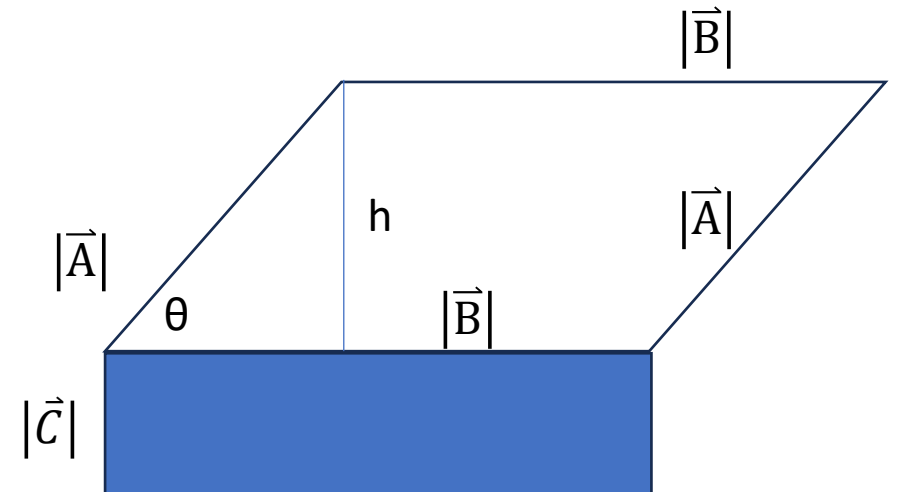
$$\text{Volume} = A \cos \theta (|\vec{B} \times \vec{C}|)$$

$$\therefore A \cdot \hat{n} = A \cos \theta$$

$$\text{Volume} = A \cdot \hat{n} (|\vec{B} \times \vec{C}|)$$

$$\text{Volume} = A \cdot (|\vec{B} \times \vec{C}| \hat{n})$$

$$\text{Volume} = A \cdot \vec{B} \times \vec{C}$$



8-Vector Algebra Operations (Vector calculus identities) Vector Derivative

Gradient of a scalar field	Divergence of a vector field	Curl of a vector field	Laplacian of a scalar field
$\vec{\nabla}$	$\vec{\nabla} \cdot$	$\vec{\nabla} \times$	$\vec{\nabla} \cdot \vec{\nabla}$
It is called "del"	Del dot	Del cross	Del dot del
$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$	$\vec{\nabla} \cdot (\vec{V}) = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (V_x \hat{i} + V_y \hat{j} + V_z \hat{k})$	$\vec{\nabla} \times (\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$	$\vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$\text{grad}(\mathbf{f}) = \vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$	$\text{div}(\vec{V}) = \vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$	$\text{Curl}(\vec{V}) = \vec{\nabla} \times \vec{V} = \hat{i} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{j} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$	$\text{Lap}(\mathbf{f}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
The function f rating (percentage) of change in a direction	The vector \vec{V} is spread out (divergences) from the point in question.	The vector \vec{V} will curl (rotate) around the point in question	The function f is changing over a small sphere centered at point.
Gradient of function f(x,y,z) in three-dimensional	The divergence of vector $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$	The curl of vector $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$	The Laplacian of function f(x,y,z) in three-dimensional

Example 8:

$$f = x^2 y z^3$$

Given a scalar field $f(x, y, z) = x^2 y z^3$ and a vector field $V = x^2 \hat{i} + yz \hat{j} - z^2 \hat{k}$, find the following at the point $P(1, 2, 1)$:

$$\vec{V} = x^2 \hat{i} + yz \hat{j} - z^2 \hat{k}$$

1. Gradient of the scalar field f .
2. Divergence of the vector field V .
3. Curl of the vector field V .
4. Laplacian of the scalar field f .

1. Gradient of f

The gradient, ∇f , is a vector that points in the direction of the greatest rate of increase of the scalar field. It's calculated as:

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$f = x^2 y z^3$$

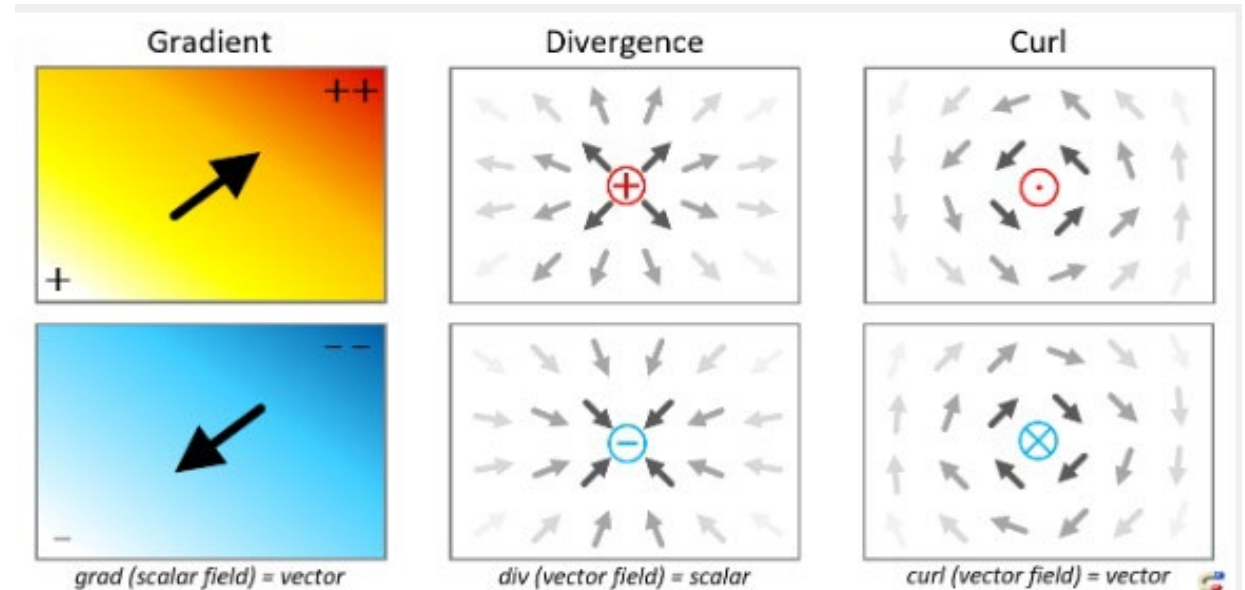
- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 y z^3) = 2x y z^3$
- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y z^3) = x^2 z^3$
- $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 y z^3) = 3x^2 y z^2$

$$\text{So, } \nabla f = (2x y z^3) \hat{i} + (x^2 z^3) \hat{j} + (3x^2 y z^2) \hat{k}$$

At point $P(1, 2, 1)$, substitute $x = 1$, $y = 2$, and $z = 1$:

$$\nabla f(1, 2, 1) = [2(1)(2)(1)^3] \hat{i} + [(1)^2(1)^3] \hat{j} + [3(1)^2(2)(1)^2] \hat{k}$$

$$\nabla f(1, 2, 1) = 4 \hat{i} + 1 \hat{j} + 6 \hat{k}$$



$$\vec{V} = x^2\hat{i} + yz\hat{j} - z^2\hat{k}$$

2. Divergence of V

The divergence, $\nabla \cdot V$, is a scalar that measures the magnitude of the vector field's source or sink at a given point. It's a "del dot" operation:

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Given $V = x^2\hat{i} + yz\hat{j} - z^2\hat{k}$, the components are $V_x = x^2$, $V_y = yz$, and $V_z = -z^2$.

- $\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x$
- $\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}(yz) = z$
- $\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}(-z^2) = -2z$

So, $\nabla \cdot V = 2x + z - 2z = 2x - z$.

At point $P(1, 2, 1)$, substitute $x = 1$ and $z = 1$:

$$\nabla \cdot V(1, 2, 1) = 2(1) - (1) = 1$$

3. Curl of V

The curl, $\nabla \times V$, is a vector that measures the tendency of a vector field to "curl" or rotate around a point. It's a "del cross" operation, best solved using a determinant. $\nabla \times V =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

With $V_x = x^2$, $V_y = yz$, and $V_z = -z^2$:

- $\frac{\partial V_z}{\partial y} = \frac{\partial}{\partial y}(-z^2) = 0$
- $\frac{\partial V_y}{\partial z} = \frac{\partial}{\partial z}(yz) = y$
- $\frac{\partial V_z}{\partial x} = \frac{\partial}{\partial x}(-z^2) = 0$
- $\frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z}(x^2) = 0$
- $\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$
- $\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}(x^2) = 0$

Substitute these into the curl formula:

$$\nabla \times V = \hat{i}(0 - y) - \hat{j}(0 - 0) + \hat{k}(0 - 0) = -y\hat{i}$$

At point $P(1, 2, 1)$, substitute $y = 2$:

$$\nabla \times V(1, 2, 1) = -2\hat{i}$$

$$\vec{V} = x^2\hat{i} + yz\hat{j} - z^2\hat{k}$$

$$\begin{aligned} \text{Curl } (\vec{V}) &= \vec{\nabla} \times \vec{V} \\ &= \hat{i} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \\ &\quad + \hat{j} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \\ &\quad + \hat{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \end{aligned}$$

4. Laplacian of f

The Laplacian, $\nabla^2 f$, is a scalar operator that provides a measure of how the average value of a scalar field differs from its value at a specific point. It's calculated as the divergence of the gradient: $\nabla^2 f = \nabla \cdot (\nabla f)$.

From Step 1, we found $\nabla f = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (3x^2yz^2)\hat{k}$.

Now, we take the divergence of this result:

$$\nabla^2 f = \frac{\partial}{\partial x}(2xyz^3) + \frac{\partial}{\partial y}(x^2z^3) + \frac{\partial}{\partial z}(3x^2yz^2)$$

- $\frac{\partial}{\partial x}(2xyz^3) = 2yz^3$
- $\frac{\partial}{\partial y}(x^2z^3) = x^2z^3$
- $\frac{\partial}{\partial z}(3x^2yz^2) = 6x^2yz$

So, $\nabla^2 f = 2yz^3 + x^2z^3 + 6x^2yz$.

At point $P(1, 2, 1)$, substitute $x = 1$, $y = 2$, and $z = 1$:

$$\nabla^2 f(1, 2, 1) = 2(2)(1)^3 + (1)^2(1)^3 + 6(1)^2(2)(1)$$

$$\nabla^2 f(1, 2, 1) = 4 + 1 + 12 = 17$$

$$f = x^2yz^3$$

$$\nabla f = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (3x^2yz^2)\hat{k}$$

Example 9:

Find the gradient for the magnitude of the position vector: $r = \sqrt{x^2 + y^2 + z^2}$.

Solution:

$$r = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\vec{\nabla}r = \frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j} + \frac{\partial r}{\partial z} \hat{k}$$

$$1- \frac{\partial r}{\partial x} = \frac{1}{2} (x^2 + y^2 + z^2)^{\frac{1}{2}-1} \left(\frac{\partial(x^2+y^2+z^2)}{\partial x} \right) = \frac{1}{2} (x^2 + y^2 + z^2)^{\frac{1}{2}-\frac{2}{2}} (2x + 0 + 0) = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} (2x) =$$

$$= \frac{1}{2} \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} (2x) = \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{1}{2} \frac{2x}{r} = \frac{x}{r}$$

$$2- \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$3- \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\vec{\nabla}r = \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} = \frac{1}{r} (x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\vec{r}}{r} = \hat{r} \equiv \hat{n}$$

"التدرج للمسافة من الأصل هو متجه الوحدة الشعاعي، ومقداره يساوي 1".

This result means that the distance from the origin increases most rapidly in the radial direction and the rate of increase is 1.

Example 10:

Calculate the divergence of the following vector functions;

1. $\vec{V}_a = x\hat{i} + y\hat{j} + z\hat{k}$

2. $\vec{V}_b = \hat{k}$

3. $\vec{V}_c = z\hat{k}$

Solution:

1. $\vec{\nabla} \cdot \vec{V}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

2. $\vec{\nabla} \cdot \vec{V}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0$

3. $\vec{\nabla} \cdot \vec{V}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1$

9-Vector Algebra Operations (Vector Integration)

Line Integral	Surface Integral	Volume Integral
Is the integral of the dot product of a vector function(\vec{F}) with a tiny (very small) displacement vector along a specific path	is the integral of the dot product of a vector function(\vec{F}) with a tiny area vector over a specific surface.	Is the integral of the product of a scalar function(ϕ) and tiny volume
$\int_a^b \vec{F} \cdot d\vec{l}$	$\int_S \vec{F} \cdot \hat{n} da$	$\vec{k} = \int_V \vec{F} dV \text{ and } J = \int_V \phi dV$
$\oint_c \vec{F} \cdot d\vec{l}$	$\oint_S \vec{F} \cdot \hat{n} da$	Where \vec{k} is a vector and J is a scalar
Line integral depends on the end points a and b	The surface integral is a scalar quantity	When the scalar function (ϕ) in the volume integral is replaced with a vector function (F), it reduces to a group of integrals of scalar functions
Line integral depends on curve along which the integration is performed.	The surface integral usually depends on the surface S	

Example 11:

Given a vector function $F = x\hat{i} + y\hat{j}$ and a scalar function $\phi(x, y, z) = xz$, find the following:

1. A **line integral** of F along a straight line path C from point $A(0, 0)$ to $B(1, 1)$.
2. A **surface integral** of F over the surface S of a unit square in the xy-plane with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$.
3. A **volume integral** of the scalar function ϕ over a cube V with sides of length 1, starting at the origin, from $x = 0$ to 1, $y = 0$ to 1, and $z = 0$ to 1.

$$\text{Line Integral} = \int_a^b \vec{F} \cdot d\vec{l} \quad \text{--- 1}$$

$$\vec{F} = x\hat{i} + y\hat{j} = \vec{r} \quad \text{--- 2}$$

$$\vec{r} = x\hat{i} + y\hat{j} \quad \text{--- 3}$$

$$\vec{r}(t) = t\hat{i} + t\hat{j} \quad \text{--- 4}$$

When we derivative eq 4

$$\frac{d\vec{r}}{dt} = (1)\hat{i} + (1)\hat{j} = \hat{i} + \hat{j} \quad \text{--- 5}$$

$$\therefore d\vec{l} = d\vec{r} = d\vec{r} \frac{dt}{dt} = \frac{d\vec{r}}{dt} dt \quad \text{--- 6}$$

$$\therefore d\vec{l} = \frac{d\vec{r}}{dt} dt \quad \text{--- 7}$$

By sub eq 5 in eq7

$$\therefore d\vec{l} = (\hat{i} + \hat{j}) dt \quad \text{--- 8}$$

By sub eq4 and eq8 in eq 1

$$\text{Line Integral} = \int_a^b \vec{F} \cdot d\vec{l} \quad \text{--- 1}$$

$$\vec{r}(t) = F = t\hat{i} + t\hat{j} \quad \text{--- 4}$$

$$d\vec{l} = (\hat{i} + \hat{j}) dt \quad \text{--- 8}$$

$$\begin{aligned} &= \int_a^b (t\hat{i} + t\hat{j}) \cdot (\hat{i} + \hat{j}) dt = \int_0^1 (t\hat{i} + t\hat{j}) \cdot (\hat{i} + \hat{j}) dt \\ &= \int_0^1 (t \cdot 1 + t \cdot 1) dt = \int_0^1 (2t) dt = [t^2]_0^1 = 1 \end{aligned}$$

2. Surface Integral

The surface integral is given by $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, da$.

The surface S is the unit square in the xy -plane. The outward normal vector for a surface in the xy -plane is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.

The differential area element is $da = dx \, dy$.

The dot product $\mathbf{F} \cdot \hat{\mathbf{n}}$ is:

$$\mathbf{F} \cdot \hat{\mathbf{n}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{k}} = x(\hat{\mathbf{i}} \cdot \hat{\mathbf{k}}) + y(\hat{\mathbf{j}} \cdot \hat{\mathbf{k}}) = x(0) + y(0) = 0$$

Now, integrate this result over the surface S :

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, da = \iint_S 0 \, dx \, dy = 0$$

The value of the surface integral is 0. This makes sense, as the vector field \mathbf{F} lies entirely in the xy -plane and has no component perpendicular to the surface.

3. Volume Integral

The volume integral is given by $J = \iiint_V \phi \, dV$.

The scalar function is $\phi = xz$.

The differential volume element is $dV = dx \, dy \, dz$.

The region V is a cube with x from 0 to 1, y from 0 to 1, and z from 0 to 1.

The integral is set up as a triple integral:

$$J = \int_0^1 \int_0^1 \int_0^1 xz \, dx \, dy \, dz$$

Integrate with respect to x first:

$$\int_0^1 xz \, dx = z \left[\frac{x^2}{2} \right]_0^1 = z \left(\frac{1^2}{2} - \frac{0^2}{2} \right) = \frac{1}{2}z$$

Now, integrate this result with respect to y :

$$\int_0^1 \left(\frac{1}{2}z \right) dy = \frac{1}{2}z [y]_0^1 = \frac{1}{2}z(1 - 0) = \frac{1}{2}z$$

Finally, integrate with respect to z :

$$\int_0^1 \frac{1}{2}z \, dz = \frac{1}{2} \left[\frac{z^2}{2} \right]_0^1 = \frac{1}{2} \left(\frac{1^2}{2} - \frac{0^2}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

The value of the volume integral is $\frac{1}{4}$.

10-Theorems

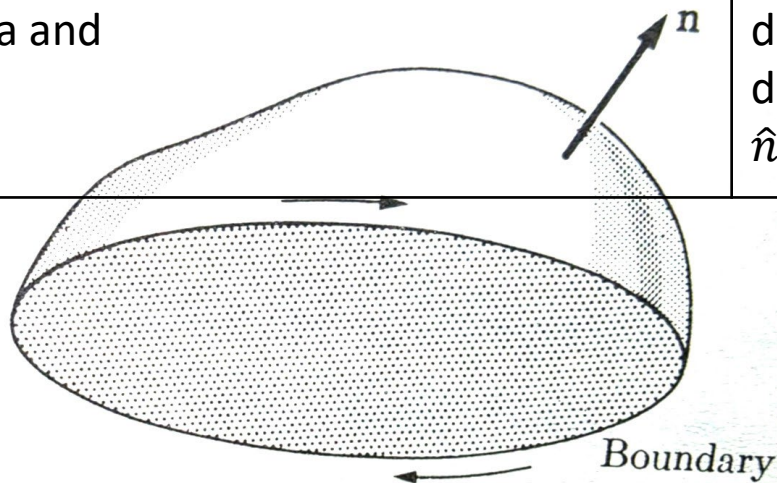
Gauss's Theorem

$$\int_V \text{Div} \vec{F} \, dV = \int_V \vec{\nabla} \cdot \vec{F} \, dV = \oint_S \vec{F} \cdot \hat{n} \, da$$

Gauss's Theorem

The integral of the divergence of a vector (F) over a volume (V) is equal to the surface (S) integral of the normal component of the vector (F) over the surface(S) bounding (V)

If S is closed surface bounding the volume V and da is very small area and \hat{n} is unit vector



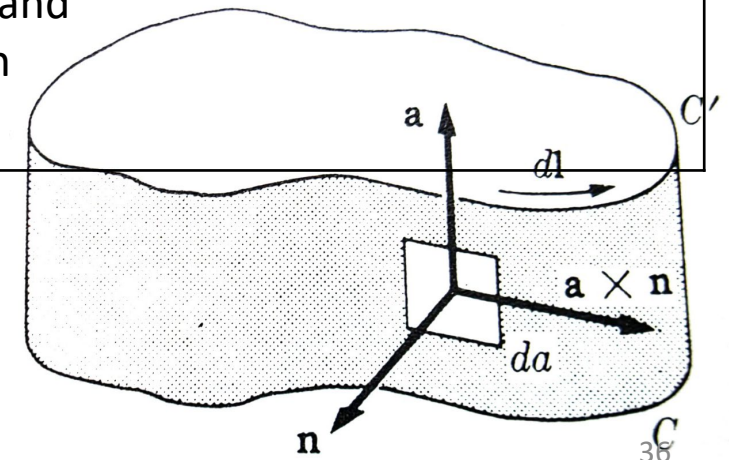
Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot \hat{n} \, da = \int_S \text{Curl} \vec{F} \cdot \hat{n} \, da$$

Stokes' Theorem

The line integral (C) of a vector field (F) around the closed curve (C) is equal to the integral of the normal component of its curl over any surface (S) bounded by the curve.

If C is closed curve around the area S da is very small area and dl is very small length \hat{n} is unit vector



10-Theorems

Gauss's Theorem

$$\int_V \text{Div} \vec{F} \, dV = \int_V \vec{\nabla} \cdot \vec{F} \, dV = \oint_S \vec{F} \cdot \hat{n} \, da$$

Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot \hat{n} \, da = \int_S \text{Curl} \vec{F} \cdot \hat{n} \, da$$

Gauss's Theorem

The integral of the divergence of a vector (F) over a volume (V) is equal to the surface (S) integral of the normal component of the vector (F) over the surface(S) bounding (V)

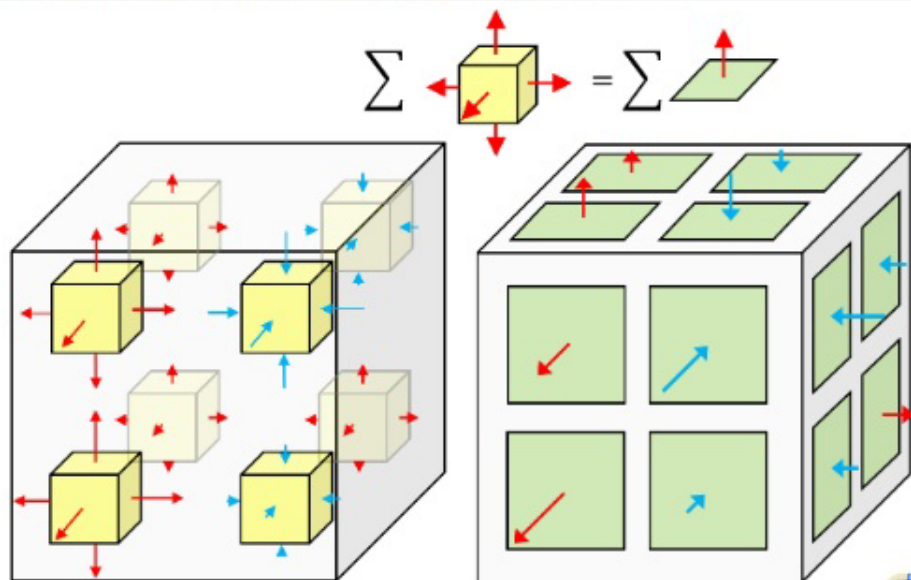
If S is closed surface bounding the volume V and
da is very small area and
 \hat{n} is unit vector

Stokes' Theorem

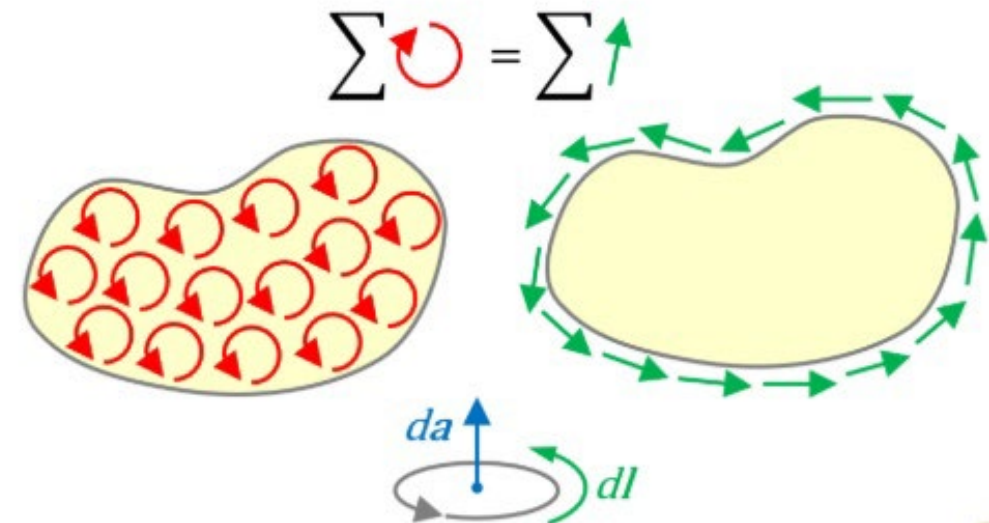
The line integral (C) of a vector field (F) around the closed curve (C) is equal to the integral of the normal component of its curl over any surface (S) bounded by the curve.

If C is closed curve around the area S
da is very small area and
dl is very small length
 \hat{n} is unit vector

Intuitive illustration of the Gauss's divergence theorem²⁹⁾:
divergence is equal to net **flux** for a given volume



Intuitive illustration of the Stokes' curl theorem³¹⁾



Problem: Verify **Gauss's Theorem** for the vector field $F = x\hat{i} + y\hat{j} + z\hat{k}$ over the surface of a cube in the first octant with sides of length 1, with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$.

Example 12:

Solution:

Gauss's Theorem states that the flux of a vector field through a closed surface is equal to the divergence of the field integrated over the enclosed volume: $\oiint_S F \cdot \hat{n} da = \iiint_V \nabla \cdot F dV$. We will calculate both sides of the equation and show that they are equal.

1. Right-Hand Side (Volume Integral)

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

First, calculate the divergence of the vector field $F = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\nabla \cdot F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

Now, integrate this divergence over the volume V of the cube, where x , y , and z all range from 0 to 1.

$$\iiint_V \nabla \cdot F dV = \iiint_V 3 dV = 3 \iiint_V dV = 3 \times (\text{Volume of the cube})$$

The volume of a unit cube is $1 \times 1 \times 1 = 1$.

$$\iiint_V \nabla \cdot F dV = 3 \times 1 = 3.$$

2. Left-Hand Side (Surface Integral)

The surface integral $\iint_S \vec{F} \cdot \hat{n} da$ is the sum of the fluxes over all six faces of the cube. Due to the symmetry of the vector field, we can observe that flux will only exist on the faces where the normal vector is aligned with the field's components.

For example, the back face is at $x = 0$, the flux is $\vec{F} \cdot (-\hat{i}) = (x\hat{i}) \cdot (-\hat{i}) = -x = 0$. By symmetry, the flux through the faces at $y = 0$ and $z = 0$ is also zero.

Let's calculate the flux for the three non-zero faces:

- **Front Face ($x = 1$):** The normal vector is $\hat{n} = \hat{i}$.

$$\int_{S_1} \vec{F} \cdot \hat{i} da = \int_0^1 \int_0^1 (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{i} dy dz = \int_0^1 \int_0^1 x dy dz$$

Since $x = 1$ on this face, the integral becomes:

$$\int_0^1 \int_0^1 1 dy dz = [y]_0^1 [z]_0^1 = (1)(1) = 1.$$

- **Top Face ($y = 1$):** The normal vector is $\hat{n} = \hat{j}$.

$$\int_{S_2} \vec{F} \cdot \hat{j} da = \int_0^1 \int_0^1 (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{j} dx dz = \int_0^1 \int_0^1 y dx dz$$

Since $y = 1$ on this face, the integral becomes:

$$\int_0^1 \int_0^1 1 dx dz = [x]_0^1 [z]_0^1 = (1)(1) = 1.$$

- **Right Face ($z = 1$):** The normal vector is $\hat{n} = \hat{k}$.

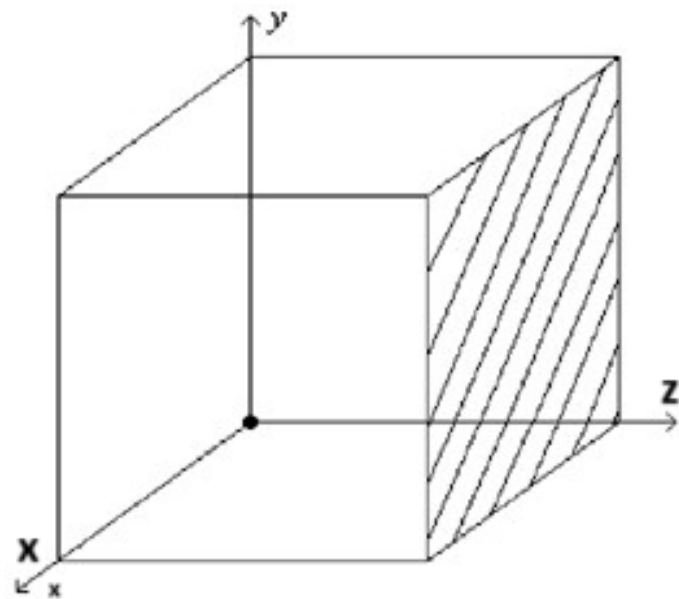
$$\int_{S_3} \vec{F} \cdot \hat{k} da = \int_0^1 \int_0^1 (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{k} dx dy = \int_0^1 \int_0^1 z dx dy$$

Since $z = 1$ on this face, the integral becomes:

$$\int_0^1 \int_0^1 1 dx dy = [x]_0^1 [y]_0^1 = (1)(1) = 1.$$

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

- **Face 2: The face where $x = 0$ (back face)**
 - The outward normal is $\hat{n} = -\hat{i}$.
 - On this face, $x = 0$, so the vector field is $\vec{F} = 0\hat{i} + y\hat{j} + z\hat{k}$.
 - $\vec{F} \cdot \hat{n} = (0\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{i}) = 0$.
 - The flux is $\int_{\text{face } x=0} 0 da = 0$.



The total flux is the sum of the fluxes from all six faces: $0 + 0 + 0 + 1 + 1 + 1 = 3$.

Since the Left-Hand Side (3) equals the Right-Hand Side (3), **Gauss's Theorem is verified.**

Example 13:

Problem: Verify **Stokes' Theorem** for the vector field $F = -y\hat{i} + x\hat{j}$ over a unit square in the xy -plane defined by the vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. The path C is the boundary of the square, traversed counter-clockwise.

Solution:

Stokes' Theorem states that the line integral of a vector field around a closed loop is equal to the surface integral of its curl over the enclosed surface: $\oint_C F \cdot dl = \iint_S (\nabla \times F) \cdot \hat{n} da$. We will calculate both sides and show they are equal.

1. Left-Hand Side (Line Integral)

$$\vec{F} = -y\hat{i} + x\hat{j}$$

The line integral is the sum of integrals along the four sides of the square.

Path C1 (from (0,0) to (1,0)):

$$y = 0, dy = 0. dl = dx\hat{i}. F = -0\hat{i} + x\hat{j} = x\hat{j}.$$

$$\int_{C_1} F \cdot dl = \int_0^1 (x\hat{j}) \cdot (dx\hat{i}) = 0.$$

Path C2 (from (1,0) to (1,1)):

$$x = 1, dx = 0. dl = dy\hat{j}. F = -y\hat{i} + 1\hat{j}.$$

$$\int_{C_2} F \cdot dl = \int_0^1 (-y\hat{i} + \hat{j}) \cdot (dy\hat{j}) = \int_0^1 1 dy = [y]_0^1 = 1.$$

Path C3 (from (1,1) to (0,1)):

$$y = 1, dy = 0. dl = dx\hat{i}. F = -1\hat{i} + x\hat{j}.$$

$$\int_{C_3} F \cdot dl = \int_1^0 (-1\hat{i} + x\hat{j}) \cdot (dx\hat{i}) = \int_1^0 -1 dx = [-x]_1^0 = 0 - (-1) = 1.$$

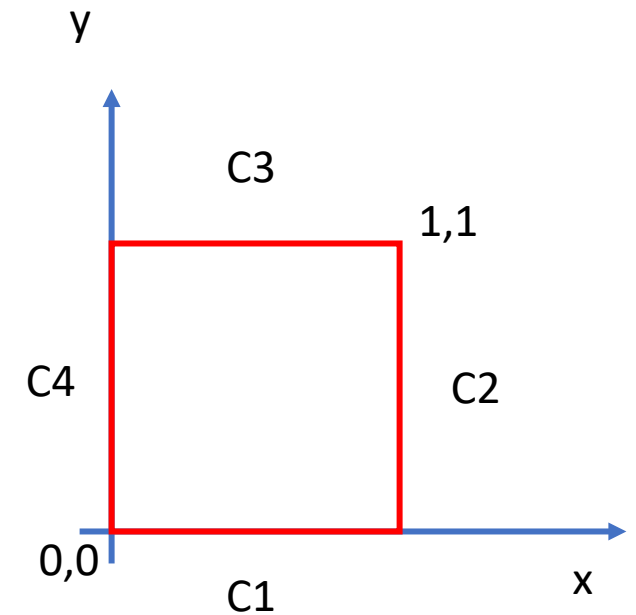
Path C4 (from (0,1) to (0,0)):

$$x = 0, dx = 0. dl = dy\hat{j}. F = -y\hat{i} + 0\hat{j} = -y\hat{i}.$$

$$\int_{C_4} F \cdot dl = \int_1^0 (-y\hat{i}) \cdot (dy\hat{j}) = 0.$$

Total line integral: $\oint_C F \cdot dl = 0 + 1 + 1 + 0 = 2.$

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot \hat{n} da$$



2. Right-Hand Side (Surface Integral)

$$\vec{F} = -y\hat{i} + x\hat{j}$$

First, calculate the curl of the vector field $F = -y\hat{i} + x\hat{j} + 0\hat{k}$.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right) + \hat{k}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)$$

$$\frac{\partial}{\partial y}(-y))$$

$$\nabla \times F = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(1 - (-1)) = 2\hat{k}.$$

The surface S is the unit square in the xy -plane. The normal vector is $\hat{n} = \hat{k}$ for counter-clockwise circulation, and the area element is $da = dx dy$.

Now, calculate the surface integral:

$$\iint_S (\nabla \times F) \cdot \hat{n} da = \iint_S (2\hat{k}) \cdot \hat{k} dx dy = \iint_S 2 dx dy$$

$$\iint_S 2 dx dy = 2 \iint_S dx dy = 2 \times (\text{Area of the square})$$

The area of a unit square is $1 \times 1 = 1$.

$$\iint_S (\nabla \times F) \cdot \hat{n} da = 2 \times 1 = 2.$$

Since the Left-Hand Side (2) equals the Right-Hand Side (2), **Stokes' Theorem is verified.**

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S (\nabla \times \vec{F}) \cdot \hat{n} da$$

11-Coordinate Systems

Cartesian (rectangular)	Cylindrical	Spherical
$-\infty < x < +\infty$ $-\infty < y < +\infty$ $-\infty < z < +\infty$	$0 \leq \rho < \infty$ <i>axial radius ρ</i> $0 \leq \phi < 2\pi$ ϕ is the azimuth $-\infty < z < +\infty$	$0 \leq r < \infty$ <i>central radius r</i> $0 \leq \theta \leq \pi$ elevation or inclination $0 \leq \phi < 2\pi$ ϕ is the azimuth
Cylindrical to Cartesian	Cartesian to Cylindrical	Cylindrical to Spherical
$x = \rho \cos\phi$ $y = \rho \sin\phi$ $z = z$	$\rho = \sqrt{x^2 + y^2}$ $\tan\phi = \frac{y}{x}$ $z = z$	$r = \sqrt{\rho^2 + z^2}$ $\phi = \phi$ $\theta = \arccos\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right) = \arctan\left(\frac{\rho}{z}\right)$
Spherical to Cartesian	Spherical to Cylindrical	Cartesian to Spherical
$x = r \sin\theta \cos\phi$ $y = r \sin\theta \sin\phi$ $z = r \cos\theta$	$\rho = r \sin\theta$ $\phi = \phi$ $z = r \cos\theta$	$r = \sqrt{x^2 + y^2 + z^2}$ $\tan\phi = \frac{y}{x}$ $\tan\theta = \left(\frac{\sqrt{x^2 + y^2}}{z}\right)$
$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$	$\vec{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{k}$	$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$
$d\vec{l} = dx \hat{i} + dy \hat{j} + dz \hat{k}$	$d\vec{l} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z}$	$d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$
$ds = dx dy = dx dz = dy dz$	$ds = d\rho dz = \rho d\rho d\phi = \rho d\phi dz$	$ds = r dr d\theta = r^2 \sin\theta d\theta d\phi = r dr \sin\theta d\phi$
$dV = dx dy dz$	$dV = \rho d\rho d\phi dz$	$dV = r^2 \sin\theta dr d\theta d\phi$

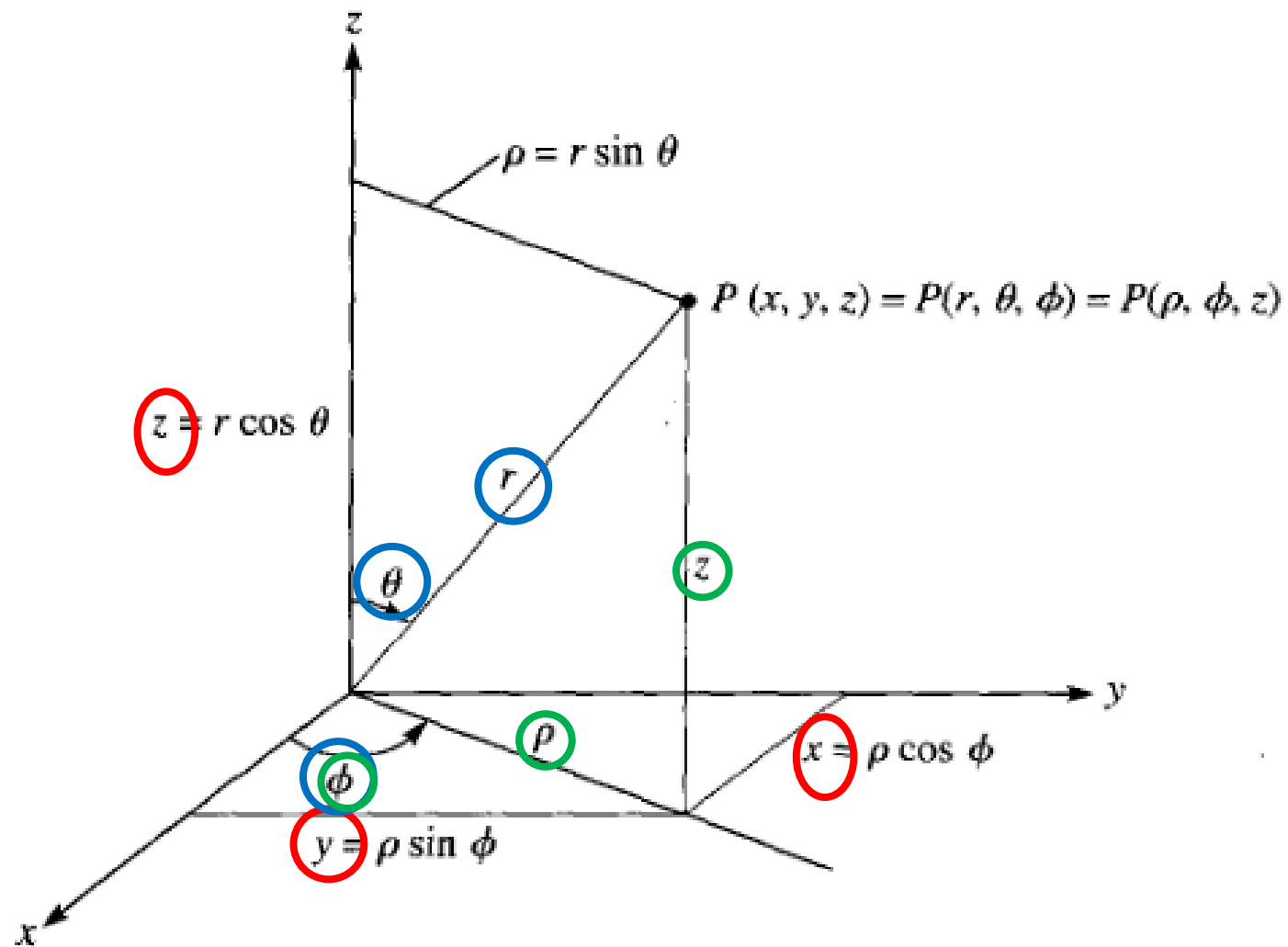
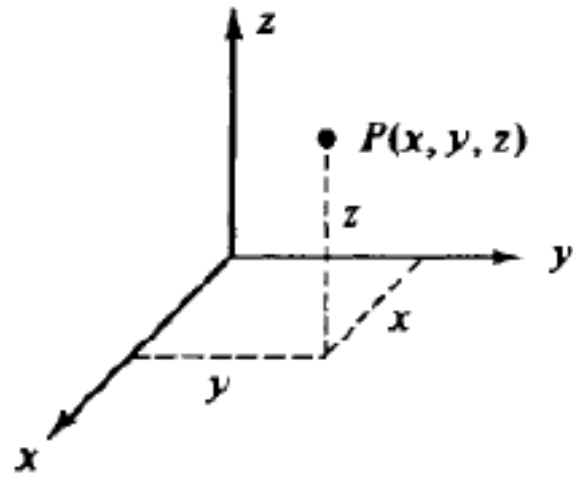
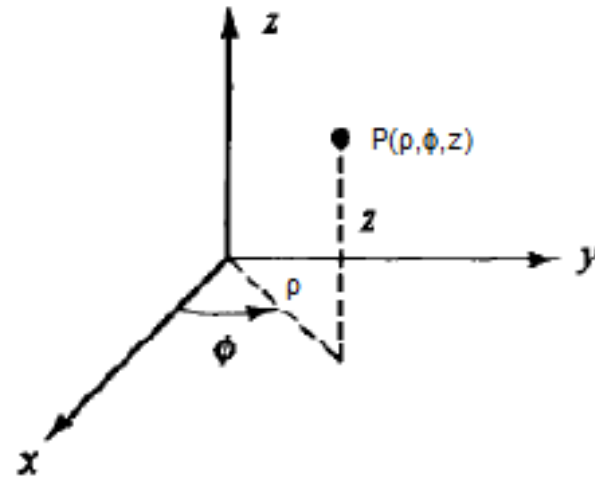


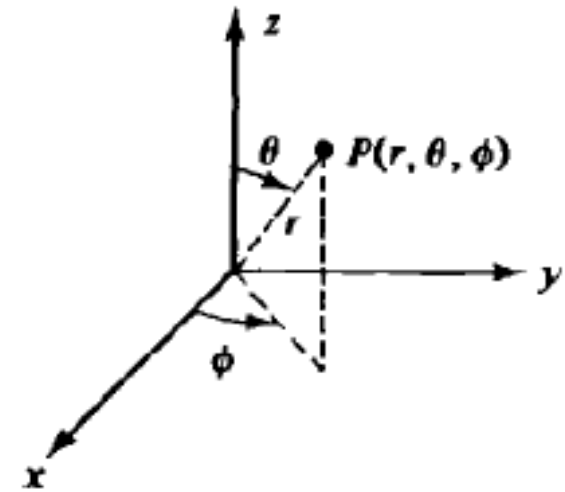
Figure 2.5 Relationships between space variables (x, y, z) , (r, θ, ϕ) , and (ρ, ϕ, z) .



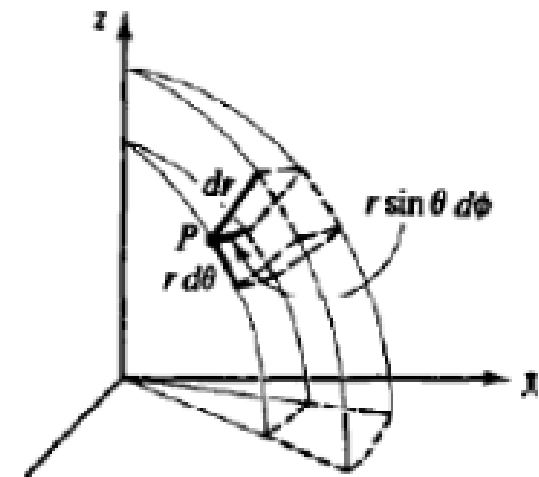
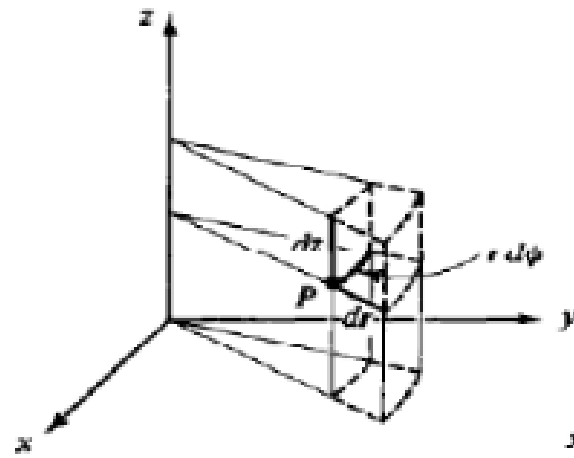
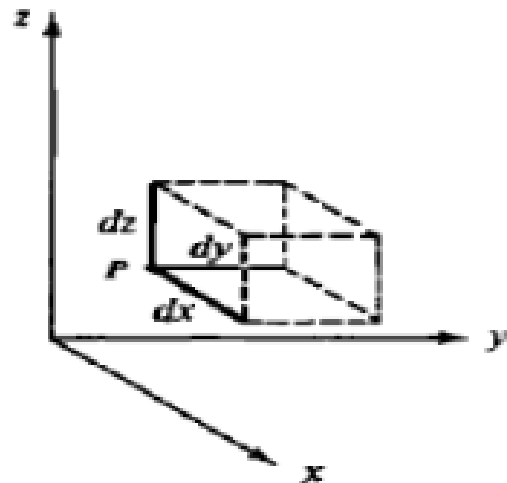
(a) Cartesian

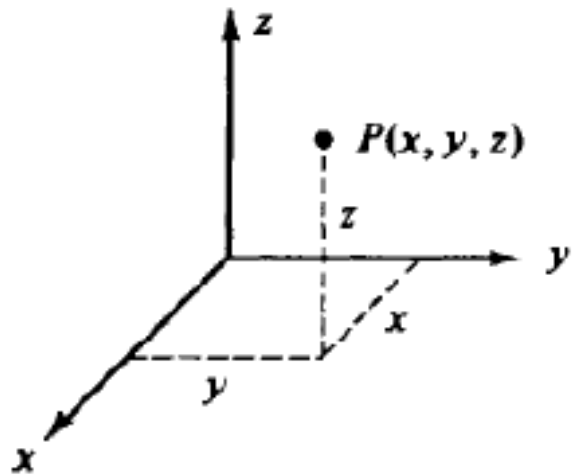


(b) Cylindrical

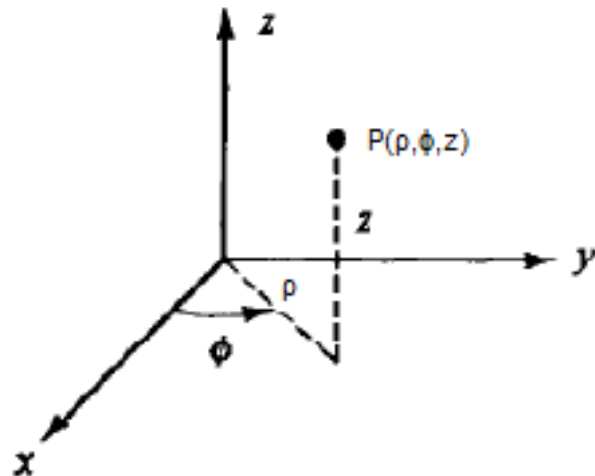


(c) Spherical

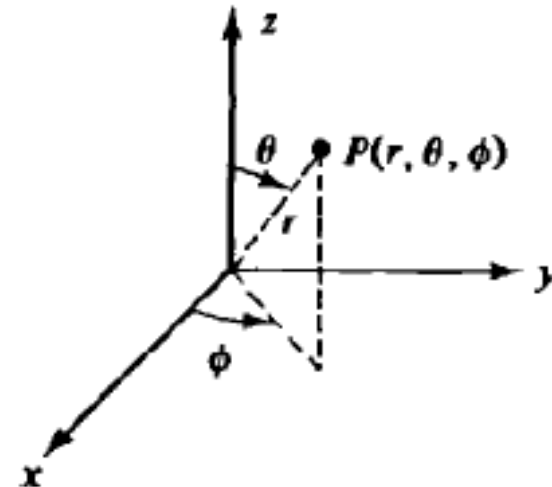




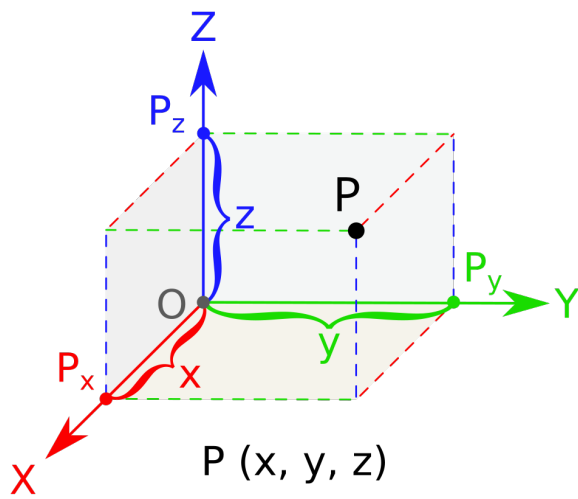
(a) Cartesian



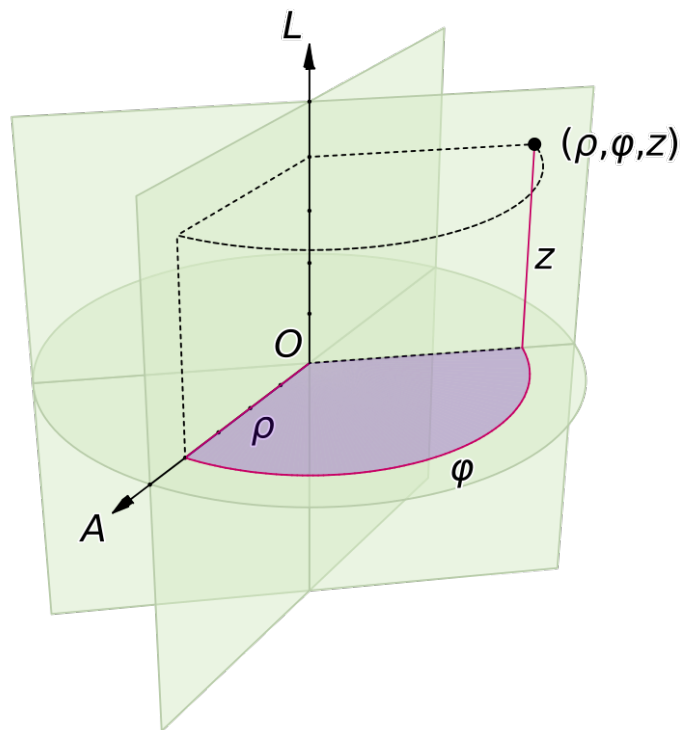
(b) Cylindrical



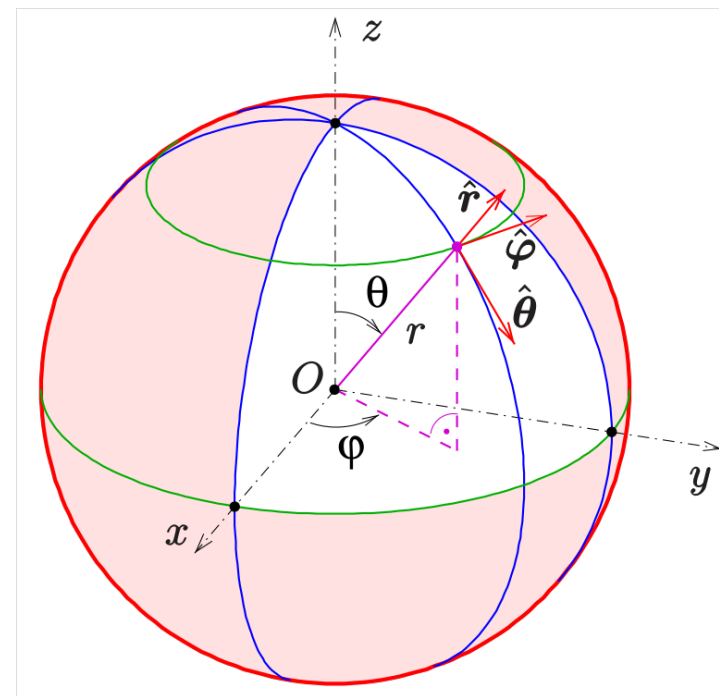
(c) Spherical



$P(x, y, z)$



(ρ, ϕ, z)



Problem 1: Cartesian Coordinates

Example 14: A particle's position is given by the vector $P = 3\hat{i} + 4\hat{j} + 5\hat{k}$. Find its distance from the origin and express the position vector in cylindrical and spherical coordinates.

Solution:

- **Distance from the origin:** The distance is the magnitude of the position vector, which is calculated using the formula for the magnitude of a vector from the first table:

$$|P| = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$$

- **Cylindrical Conversion:** Use the formulas from the table for converting Cartesian to Cylindrical coordinates: $\rho = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$, and $z = z$.

$$\rho = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\phi = \arctan(4/3) \approx 0.927 \text{ radians}$$

$$z = 5$$

The cylindrical coordinates are $(\rho, \phi, z) = (5, \arctan(4/3), 5)$.

- **Spherical Conversion:** Use the formulas from the table for converting Cartesian to

Spherical coordinates: $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$, and $\phi = \arctan(y/x)$

$$r = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

$$\theta = \arctan\left(\frac{\sqrt{3^2 + 4^2}}{5}\right) = \arctan\left(\frac{5}{5}\right) = \arctan(1) = \pi/4$$

$$\phi = \arctan(4/3) \approx 0.927 \text{ radians}$$

The spherical coordinates are $(r, \theta, \phi) = (5\sqrt{2}, \pi/4, \arctan(4/3))$.

Example 15: ρ : Cylindrical Coordinates

Problem: A cylindrical tank has a radius of 3 m and a height of 5 m. Use a volume integral in cylindrical coordinates to calculate the volume of the tank.

Solution:

- **Volume Integral Setup:** The volume element in cylindrical coordinates is $dV = \rho d\rho d\phi dz$.

The limits of integration for the cylindrical tank are:

- Radius ρ goes from 0 to 3.
- Azimuthal angle ϕ goes from 0 to 2π .
- Height z goes from 0 to 5.

- **Calculate the Integral:**

$$V = \iiint_V dV = \int_0^5 \int_0^{2\pi} \int_0^3 \rho d\rho d\phi dz$$

Integrate with respect to ρ :

$$\int_0^3 \rho d\rho = \left[\frac{\rho^2}{2}\right]_0^3 = \frac{3^2}{2} - 0 = \frac{9}{2}$$

Integrate with respect to ϕ :

$$\int_0^{2\pi} \left(\frac{9}{2}\right) d\phi = \frac{9}{2} [\phi]_0^{2\pi} = \frac{9}{2} (2\pi - 0) = 9\pi$$

Integrate with respect to z :

$$\int_0^5 (9\pi) dz = 9\pi [z]_0^5 = 9\pi (5 - 0) = 45\pi$$

The volume of the cylindrical tank is 45π cubic meters.

Example 16: : Spherical Coordinates

Problem: A scalar field is given by $\phi = x^2 + y^2 + z^2$. Find the value of the volume integral $\iiint_V \phi dV$ over a spherical shell with an inner radius of 2 and an outer radius of 3.

Solution:

- **Convert to Spherical Coordinates:** The scalar function $x^2 + y^2 + z^2$ is simply r^2 in spherical coordinates. The volume element is $dV = r^2 \sin \theta dr d\theta d\phi$.

The limits of integration for the spherical shell are:

- Radius r goes from 2 to 3.
- Polar angle θ goes from 0 to π .
- Azimuthal angle ϕ goes from 0 to 2π .

- **Calculate the Integral:**

$$\iiint_V \phi dV = \int_0^{2\pi} \int_0^\pi \int_2^3 (r^2)(r^2 \sin \theta) dr d\theta d\phi$$

$$\iiint_V \phi dV = \int_0^{2\pi} \int_0^\pi \int_2^3 r^4 \sin \theta dr d\theta d\phi$$

Integrate with respect to r :

$$\int_2^3 r^4 dr = \left[\frac{r^5}{5} \right]_2^3 = \frac{3^5}{5} - \frac{2^5}{5} = \frac{243-32}{5} = \frac{211}{5}$$

Integrate with respect to θ :

$$\int_0^\pi \sin \theta d\theta = [-\cos \theta]_0^\pi = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2$$

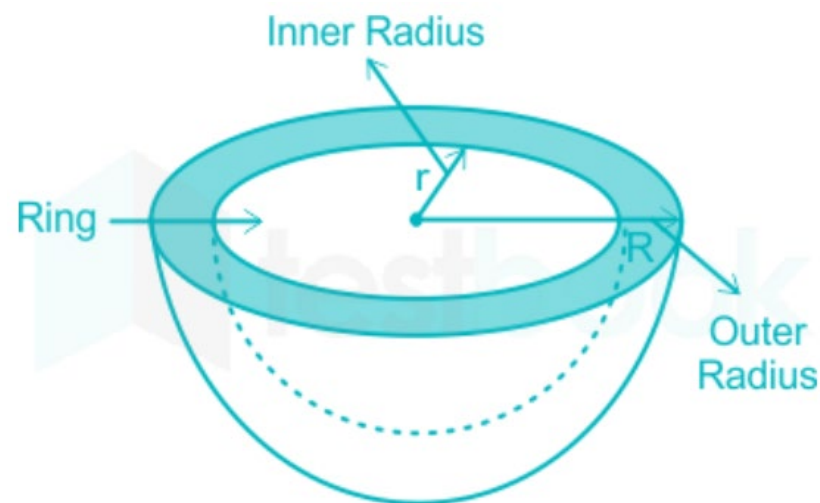
Integrate with respect to ϕ :

$$\int_0^{2\pi} (2) d\phi = 2[\phi]_0^{2\pi} = 2(2\pi - 0) = 4\pi$$

Finally, combine the results:

$$\left(\frac{211}{5} \right) \times (2) \times (4\pi) = \frac{1688\pi}{5}$$

The value of the volume integral is $\frac{1688\pi}{5}$.



Example 17: A point has Cartesian coordinates $(x, y, z) = (-1, 2, 3)$. Convert these coordinates to cylindrical and spherical coordinates.

Solution:

1. Convert to Cylindrical Coordinates

We'll use the formulas $\rho = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$, and $z = z$.

- **Calculate ρ :**
$$\rho = \sqrt{(-1)^2 + (2)^2} = \sqrt{1 + 4} = \sqrt{5}$$
- **Calculate ϕ :** Since the point is in the second quadrant ($x < 0, y > 0$), we must adjust the angle returned by \arctan .
$$\phi = \arctan\left(\frac{2}{-1}\right) = \arctan(-2) \approx -1.107 \text{ rad.}$$

To get the correct angle in the second quadrant, we add π :
$$\phi = \pi + \arctan(-2) \approx \pi - 1.107 \approx 2.034 \text{ rad.}$$
- **Calculate z :**
$$z = 3$$

The cylindrical coordinates are $(\sqrt{5}, \pi - \arctan(2), 3)$.

2. Convert to Spherical Coordinates

We'll use the formulas $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$, and $\phi = \arctan(y/x)$ with the same quadrant adjustment as above.

- **Calculate r :**
$$r = \sqrt{(-1)^2 + (2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$
- **Calculate θ :**
$$\theta = \arctan\left(\frac{\sqrt{(-1)^2 + (2)^2}}{3}\right) = \arctan\left(\frac{\sqrt{5}}{3}\right) \approx 0.643 \text{ rad.}$$
- **Calculate ϕ :** This is the same as the cylindrical angle.
$$\phi = \pi + \arctan(-2) \approx 2.034 \text{ rad.}$$

The spherical coordinates are $(\sqrt{14}, \arctan(\frac{\sqrt{5}}{3}), \pi - \arctan(2))$.

Example 18: A point has cylindrical coordinates $(\rho, \phi, z) = (5, \pi/4, -2)$. Convert these to Cartesian and spherical coordinates.

Solution:

1. Convert to Cartesian Coordinates

We'll use the formulas $x = \rho \cos \phi$, $y = \rho \sin \phi$, and $z = z$.

- **Calculate x :**
 $x = 5 \cos(\pi/4) = 5\left(\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{2}$
- **Calculate y :**
 $y = 5 \sin(\pi/4) = 5\left(\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{2}$
- **Calculate z :**
 $z = -2$

The Cartesian coordinates are $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}, -2\right)$.

2. Convert to Spherical Coordinates

We'll use the formulas $r = \sqrt{\rho^2 + z^2}$, $\theta = \arctan(\rho/z)$, and $\phi = \phi$.

- **Calculate r :**
 $r = \sqrt{5^2 + (-2)^2} = \sqrt{25 + 4} = \sqrt{29}$
- **Calculate θ :** Since z is negative, we add π to the angle returned by $\arctan(\rho/z)$.
 $\theta = \arctan\left(\frac{5}{-2}\right) = \arctan(-2.5) \approx -1.19 \text{ rad.}$
 $\theta = \pi + \arctan(-2.5) \approx \pi - 1.19 \approx 1.951 \text{ rad.}$
- **Calculate ϕ :**
 $\phi = \pi/4$

The spherical coordinates are $(\sqrt{29}, \pi + \arctan(-2.5), \pi/4)$.

Example 19: A point has spherical coordinates $(r, \theta, \phi) = (4, \pi/3, 5\pi/6)$. Convert these to Cartesian and cylindrical coordinates.

Solution:

1. Convert to Cartesian Coordinates

We'll use the formulas $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, and $z = r \cos \theta$.

- **Calculate x :**
 $x = 4 \sin(\pi/3) \cos(5\pi/6) = 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = 4\left(-\frac{3}{4}\right) = -3$
- **Calculate y :**
 $y = 4 \sin(\pi/3) \sin(5\pi/6) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = 4\left(\frac{\sqrt{3}}{4}\right) = \sqrt{3}$
- **Calculate z :**
 $z = 4 \cos(\pi/3) = 4\left(\frac{1}{2}\right) = 2$

The Cartesian coordinates are $(-3, \sqrt{3}, 2)$.

2. Convert to Cylindrical Coordinates

We'll use the formulas $\rho = r \sin \theta$, $\phi = \phi$, and $z = r \cos \theta$.

- **Calculate ρ :**
 $\rho = 4 \sin(\pi/3) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$
- **Calculate ϕ :**
 $\phi = 5\pi/6$
- **Calculate z :**
 $z = 4 \cos(\pi/3) = 4\left(\frac{1}{2}\right) = 2$

The cylindrical coordinates are $(2\sqrt{3}, 5\pi/6, 2)$.

12-Orthogonal curvilinear coordinate equations

الإحداثيات المتعامدة المنحنية

Orthogonal curvilinear coordinate equations describe physical phenomena in coordinate systems where the axes are not fixed and perpendicular, but instead curve and are always mutually orthogonal at every point. Key equations include the transformation from Cartesian coordinates,

where position vector $r = (x, y, z)$ becomes $r = (u_1, u_2, u_3)$;

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the line element, $|dr| = h_1 du_1 + h_2 du_2 + h_3 du_3$, with scale factors h_i that vary with position

Curvilinear	Cartesian Coordinate	Cylindrical Coordinate	Spherical Coordinate
u_1	x	ρ	r
u_2	y	Φ	θ
u_3	z	z	Φ
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$r \sin \theta$
\hat{e}_1	\hat{i}	$\hat{\rho}$	\hat{r}
\hat{e}_2	\hat{j}	$\hat{\Phi}$	$\hat{\theta}$
\hat{e}_3	\hat{k}	\hat{z}	$\hat{\Phi}$

1. Transformation Equations u_i

The core of a curvilinear system is the transformation from Cartesian (x, y, z) to the curvilinear coordinates (u₁, u₂, u₃):

$$x = x(u_1, u_2, u_3)$$

$$y = y(u_1, u_2, u_3)$$

$$z = z(u_1, u_2, u_3)$$

"الإحداثيات المتعامدة المنحنية هي مثل استخدام شبكة مرنة من الخطوط المنحنية تلائم شكل الجسم المدروس (كروي، أسطواني، إلخ)، مع ضمان أن هذه الخطوط تتقاطع دائماً بشكل عمودي (بزاوية قائمة) لتسهيل إجراء الحسابات عليها."

• الإحداثيات الكارتيزية: مثل السير في مدينة مبانيها على شكل "شبكة مربعة". تتحرك شمال/جنوب أو شرق/غرب في خطوط مستقيمة.

• الإحداثيات المنحنية المتعامدة: مثل السير في مدينة دائرية (محيط الكعبة المشرفة). أنت تتحرك إما "للخارج وللداخل" (نصف القطر)، أو "حول الدائرة" (الزاوية)، أو "لأعلى ولأسفل". هذه الاتجاهات متعامدة مع بعضها في أي نقطة تقف فيها، رغم أن الطرق نفسها منحنية.

2. Unit Tangent Vectors e_i

These are the non-Cartesian base vectors, which point along the coordinate curves and are orthogonal to each other in an orthogonal system.

$$e_i = \frac{1}{h_i} \frac{\partial r}{\partial u_i}$$

3-Scale Factors h_i

These represent the magnitude of the change in the position vector for a unit change in the coordinate,

$$h_i = \left| \frac{\partial r}{\partial u_i} \right|.$$

In Cartesian coordinates, $h_i = 1$.

4. Operators in Orthogonal Curvilinear Coordinates

These operators are crucial for physics and engineering problems solved in non-Cartesian systems:

$$\text{Gradient, } \vec{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} e_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} e_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} e_3$$

$$\text{Divergence, } \nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (h_2 h_3 V_1)}{\partial u_1} + \frac{\partial (h_3 h_2 V_2)}{\partial u_2} + \frac{\partial (h_1 h_2 V_3)}{\partial u_3} \right]$$

$$\text{Curl, } \nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[e_1 (h_2 V_3 - h_3 V_2) + e_2 (h_3 V_1 - h_1 V_3) + e_3 (h_1 V_2 - h_2 V_1) \right]$$

$$\text{Laplacian, } \nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right]$$

The Line Element	$dL = h_1 du_1 e_1 + h_2 du_2 e_2 + h_3 du_3 e_3$
The Surface Element	$dS = h_1 h_2 du_1 du_2 = h_1 h_3 du_1 du_3 = h_2 h_3 du_2 du_3$
The Volume Element	$dV = h_1 h_2 h_3 du_1 du_2 du_3$

Example 20:

What will happen when a point $p(x, y, z)$ is expanded to $(x+dx, y+dy, z+dz)$ or $(r+dr, \phi+d\phi, z+dz)$ or $(r+dr, \theta+d\theta, \phi+d\phi)$?

Answer:

As shown in the figures;

- 1) A differential volume dV is formed;

$$dV = dx dy dz \quad (\text{Cartesian})$$

$$dV = \rho d\rho d\phi dz \quad (\text{Cylindrical})$$

$$dV = r^2 \sin\theta dr d\theta d\phi \quad (\text{Spherical})$$

- 1) A differential area dS is formed;

$$ds = dx dy, = dx dz, = dy dz \quad (\text{Cartesian})$$

$$ds = \rho d\rho dz, = \rho d\rho d\phi, = \rho d\phi dz \quad (\text{Cylindrical})$$

$$ds = r dr d\theta, = r^2 \sin\theta d\theta d\phi, = r \sin\theta dr d\phi \quad (\text{Spherical})$$

- 1) A differential line $d\ell$ is formed;

$$d\vec{\ell} = dx\hat{i} + dy\hat{j} + dz\hat{k} \quad (\text{Cartesian})$$

$$d\vec{\ell} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z} \quad (\text{Cylindrical})$$

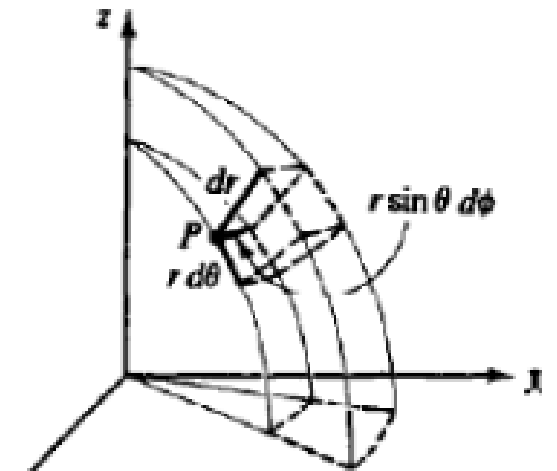
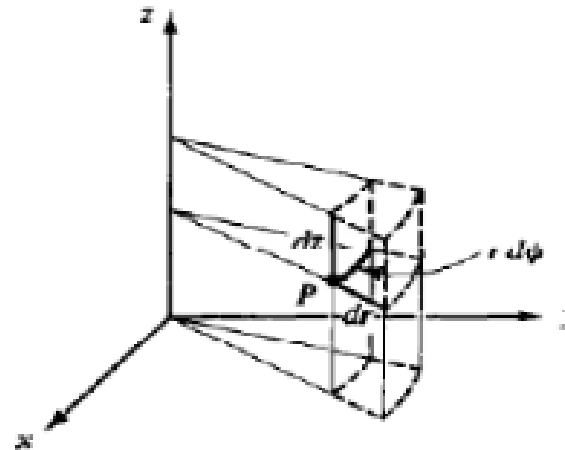
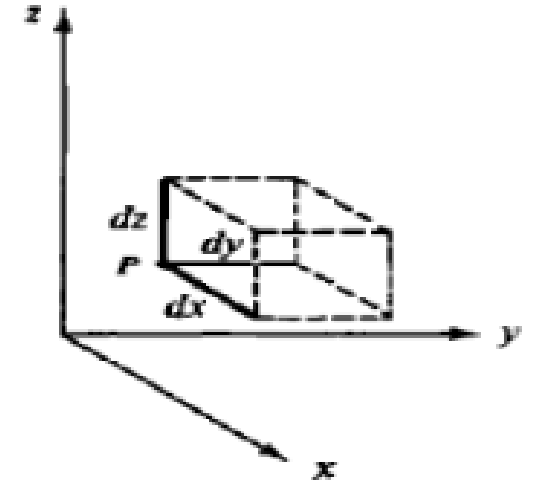
$$d\vec{\ell} = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi} \quad (\text{Spherical})$$

However;

$$dl^2 = dx^2 + dy^2 + dz^2 \quad (\text{Cartesian})$$

$$dl^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \quad (\text{Cylindrical})$$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (\text{Spherical})$$



Example 21

We'll use the point P with cylindrical coordinates:

- $\rho = 3$ (radial distance)
- $\varphi = 60^\circ = \frac{\pi}{3}$ radians (angle)
- $z = 4$

Find

1. Transformation to Cartesian Coordinates
2. Scale Factors h_i
3. Unit Tangent Vectors e_i at Point P
4. The Line Element $d\vec{L}$
5. The Volume Element dV

1. Transformation to Cartesian Coordinates

The transformation equations are:

- $x = \rho \cos \varphi$
- $y = \rho \sin \varphi$
- $z = z$

Let's calculate the Cartesian coordinates of point P :

- $x = 3 \times \cos(60^\circ) = 3 \times 0.5 = 1.5$
- $y = 3 \times \sin(60^\circ) = 3 \times \frac{\sqrt{3}}{2} \approx 3 \times 0.866 = 2.598$
- $z = 4$

So, the Cartesian coordinate of point P is approximately $(1.5, 2.598, 4)$.

2. Scale Factors h_i

As derived from the general formulas:

- $h_\rho = 1$
- $h_\varphi = \rho$
- $h_z = 1$

At our specific point $P(3, 60^\circ, 4)$:

- $h_\rho = 1$
- $h_\varphi = 3$
- $h_z = 1$

3. Unit Tangent Vectors e_i at Point P

In Cartesian coordinates: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In cylindrical coordinates: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, $z = z$

So: $\vec{r} = (\rho \cos \varphi)\hat{i} + (\rho \sin \varphi)\hat{j} + z\hat{k}$

To **Find tangent vectors (not yet unit vectors)**

We take partial derivatives of \vec{r} with respect to each coordinate:

or ρ -direction: $\frac{\partial \vec{r}}{\partial \rho} = (\cos \varphi)\hat{i} + (\sin \varphi)\hat{j} + 0\hat{k}$

For φ -direction: $\frac{\partial \vec{r}}{\partial \varphi} = (-\rho \sin \varphi)\hat{i} + (\rho \cos \varphi)\hat{j} + 0\hat{k}$

For z-direction: $\frac{\partial \vec{r}}{\partial z} = 0\hat{i} + 0\hat{j} + 1\hat{k}$

To **convert to UNIT vectors**

We divide each tangent vector by its magnitude:

for e_ρ

Vector: $(\cos \varphi)\hat{i} + (\sin \varphi)\hat{j}$

Magnitude: $\sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1$

So $e_\rho = (\cos \varphi)\hat{i} + (\sin \varphi)\hat{j}$

for e_φ

Vector: $(-\rho \sin \varphi)\hat{i} + (\rho \cos \varphi)\hat{j}$

Magnitude: $\sqrt{\rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi} = \rho$

So $e_\varphi = \frac{1}{\rho} [(-\rho \sin \varphi)\hat{i} + (\rho \cos \varphi)\hat{j}] = (-\sin \varphi)\hat{i} + (\cos \varphi)\hat{j}$

3. Unit Tangent Vectors e_i at Point P

These vectors depend on the location. Let's calculate them for point P .

- $\hat{e}_\rho = (\cos \varphi) \hat{i} + (\sin \varphi) \hat{j}$
 - $\hat{e}_\rho = (0.5) \hat{i} + (0.866) \hat{j}$
- $\hat{e}_\varphi = (-\sin \varphi) \hat{i} + (\cos \varphi) \hat{j}$
 - $\hat{e}_\varphi = (-0.866) \hat{i} + (0.5) \hat{j}$
- $\hat{e}_z = \hat{k}$

These three vectors form an orthogonal basis at point P . You can verify they are perpendicular by showing their dot products are zero (e.g., $\hat{e}_\rho \cdot \hat{e}_\varphi = (0.5)(-0.866) + (0.866)(0.5) = 0$).

4. The Line Element $d\vec{L}$

The general form is:

$$d\vec{L} = h_\rho d\rho \hat{e}_\rho + h_\varphi d\varphi \hat{e}_\varphi + h_z dz \hat{e}_z$$

At our specific point P:

$$d\vec{L} = (1) d\rho \hat{e}_\rho + (3) d\varphi \hat{e}_\varphi + (1) dz \hat{e}_z$$

This means:

- A step of $d\rho = 0.1$ m in the radial direction covers a distance of **0.1 m**.
- A step of $d\varphi = 0.1$ rad in the angular direction covers a distance of **$3 \times 0.1 = 0.3$ m**.
- A step of $dz = 0.1$ m in the z-direction covers a distance of **0.1 m**.

5. The Volume Element dV

The general form is:

$$dV = h_\rho h_\varphi h_z d\rho d\varphi dz$$

At our specific point P:

$$dV = (1)(3)(1) d\rho d\varphi dz = 3 d\rho d\varphi dz$$

If we have a small "curvilinear box" with sides $d\rho = 0.1$, $d\varphi = 0.1$, $dz = 0.1$, its volume is:

$$dV = 3 \times 0.1 \times 0.1 \times 0.1 = 0.003 \text{ cubic meters.}$$

Q.1:

Show that the vector directed from $M(x_1, y_1, z_1)$ to $N(x_2, y_2, z_2)$ in the figure below is given by: $(x_2 - x_1)\hat{i} - (y_2 - y_1)\hat{j} - (z_2 - z_1)\hat{k}$

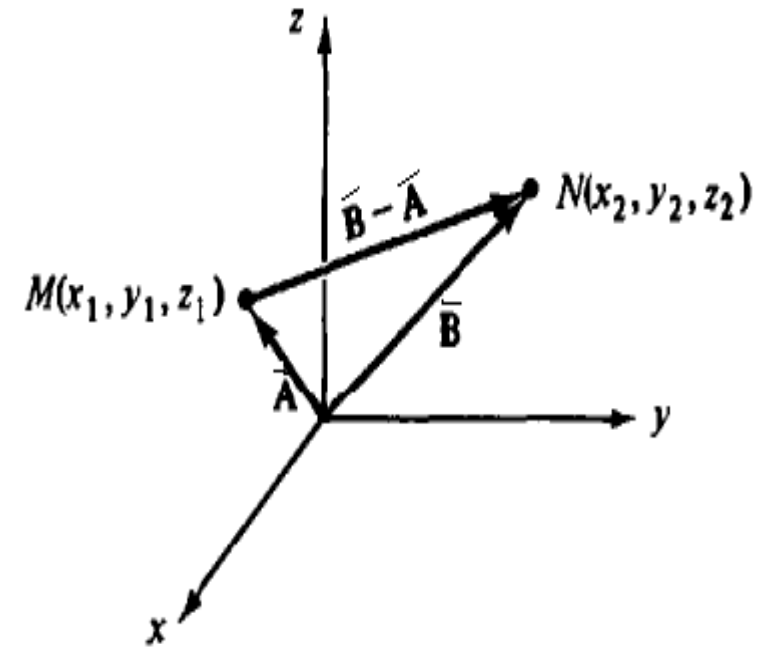
Solution:

M and N are the two locations (in Cartesian coordinate) of those points in three dimensions. The two *position vectors* for them can be denoted by \vec{A} and \vec{B} , respectively. Then:

$\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ is the position vector from the origin to the M and :

$\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ is the position vector from the origin to the N. So the vector directed from M to N is given by the subtraction of \vec{A} from \vec{B} which gives:

$$\vec{B} - \vec{A} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



Q.2:

Find the vector \vec{A} directed from $(2, -4, 1)$ to $(0, -2, 0)$ in Cartesian coordinates and find the unit vector along \vec{A} .

Solution:

The required position vector is:

$$\vec{A} = (0 - 2)\hat{i} + [-2 - (-4)]\hat{j} + (0 - 1)\hat{k} = -2\hat{i} + 2\hat{j} - \hat{k}$$

$$|\vec{A}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$\rightarrow \hat{A} = \frac{\vec{A}}{|\vec{A}|} = -\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

Q.3:

Find the distance between $(5, 3\pi/2, 0)$, and $(5, \pi/2, 10)$ in cylindrical coordinates.

Solution:

Let: $M = (5, 3\pi/2, 0)$, and $N = (5, \pi/2, 10)$.

The transformation equations from cylindrical to Cartesian coordinates are given as; $x = r \cos\phi$, $y = r \sin\phi$, $z = z$.

Thus for point M: $x = 5 \cos\left(\frac{3\pi}{2}\right) = 0$, $y = 5 \sin\left(\frac{3\pi}{2}\right) = -5$, $z = 0$. So M will be $(0, -5, 0)$ in cylindrical coordinate.

Also, for N: $x = 5 \cos\left(\frac{\pi}{2}\right) = 0$, $y = 5 \sin\left(\frac{\pi}{2}\right) = 5$, $z = 10$, thus N will be $(0, 5, 10)$ in cylindrical coordinate. Accordingly,

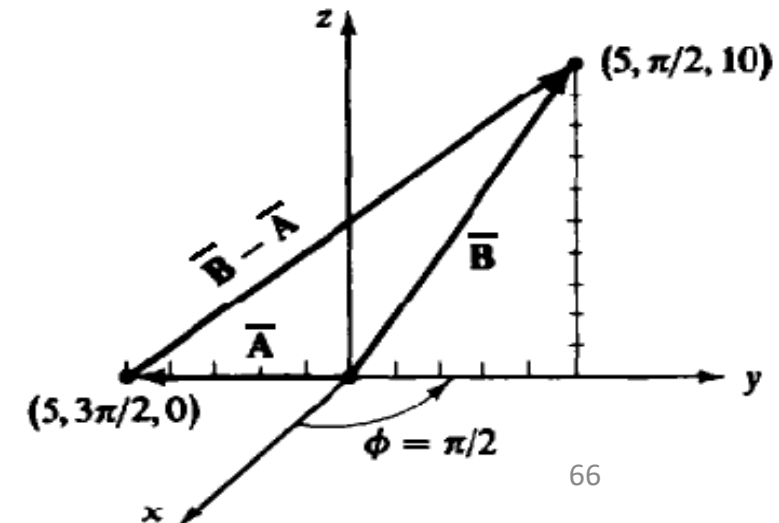
the position vector from the origin to M is: $\vec{A} = -5\hat{j}$ and the position vector from the origin to N is: $\vec{B} = 5\hat{j} + 10\hat{k}$.

- The position vector from M to N in Cartesian coordinate then calculates as:

$$\vec{B} - \vec{A} = 5\hat{j} + 10\hat{k} - (-5\hat{j}) = 10\hat{j} + 10\hat{k},$$

- Then, the required distance between the two points is;

$$|\vec{A} - \vec{B}| = 10\sqrt{2}$$



Q.4:

Show that the two vectors: $\vec{A} = 4\hat{i} - 2\hat{j} - \hat{k}$ and $\vec{B} = \hat{i} + 4\hat{j} - 4\hat{k}$ are perpendicular.

Solution:

For the two vectors: $\vec{A} = 4\hat{i} - 2\hat{j} - \hat{k}$ and $\vec{B} = \hat{i} + 4\hat{j} - 4\hat{k}$ we have:

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos\theta = A_x \cdot B_x + A_y B_y + A_z B_z = 4 - 8 + 4 = 0$$

Since the dot product contains $\cos\theta$, a dot product of zero from any two nonzero vectors implies that the two vectors are perpendicular ($\theta = 90^\circ$), i.e:

$$|\vec{A}| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{21}, |\vec{B}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{33}$$

$$\therefore \vec{A} \cdot \vec{B} = \sqrt{21} \cdot \sqrt{33} \cos\theta = 0 \xrightarrow{\text{yields}} \theta = 90^\circ.$$

Q.5: Given; $\vec{A} = 2\hat{i} + 4\hat{j}$ and $\vec{B} = 6\hat{i} - 4\hat{k}$, find the smaller angle between them using: a) the cross product and b) the dot product.

Solution:

$$(a) \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 0 \\ 0 & 6 & -4 \end{vmatrix} = -16\hat{i} + 8\hat{j} + 12\hat{k}$$

$$|\vec{A}| = \sqrt{(2)^2 + (4)^2 + (0)^2} = 4.47$$

$$|\vec{B}| = \sqrt{(0)^2 + (6)^2 + (4)^2} = 7.21$$

$$|\vec{A} \times \vec{B}| = \sqrt{(-16)^2 + (8)^2 + (12)^2} = 21.54, \text{ then;}$$

$$\text{since: } |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

$$\sin\theta = \frac{21.54}{(4.47)(7.21)} = 0.668 \text{ or } \theta = 41.9^\circ$$

$$(b) \vec{A} \cdot \vec{B} = (2)(0) + (4)(6) + (0)(-4) = 24$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{24}{(4.47)(7.21)} = 0.745 \text{ or } \theta = 41.9^\circ$$

Q.6:

Given; $\vec{F} = (y - 1)\hat{i} + 2x\hat{j}$, find the vector at (2,2,1) and its projection on B, where $\vec{B} = 5\hat{i} - \hat{j} + 2\hat{k}$

Solution:

$$\begin{aligned} F(2,2,1) &= (2 - 1)\hat{i} + (2)(2)\hat{j} \\ &= \hat{i} + 4\hat{j} \end{aligned}$$

Note: As indicated in fig., the projection of one vector (\vec{A}) on a second vector (\vec{B}) is obtained by expressing the unite vector in the direction of the second vector (u_B) and taking the dot product with first vector (\vec{A}).

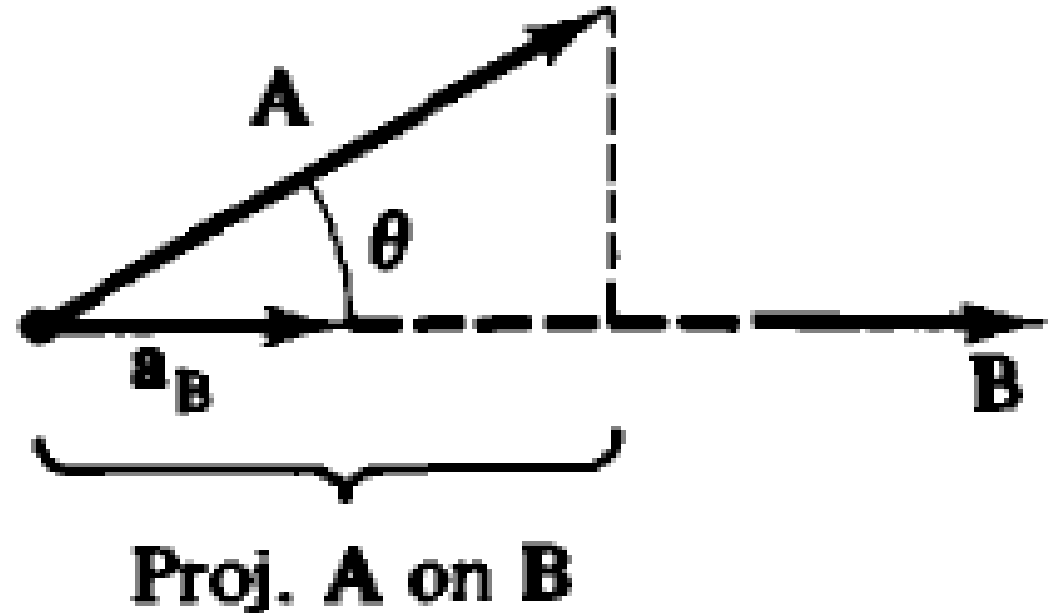
$$\text{Proj. } \vec{A} \text{ on } \vec{B} = \vec{A} \cdot a_B = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$$

$$\vec{A} = \vec{F} = (y - 1)\hat{i} + 2x\hat{j}$$

$$|\vec{B}| = B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{30}$$

Thus, at (2,2,1)

$$\text{Proj. } F \text{ on } \vec{B} = \frac{\vec{F} \cdot \vec{B}}{|\vec{B}|} = \frac{(1)(5) + (4)(-1) + (0)(2)}{\sqrt{30}} = \frac{1}{\sqrt{30}}$$



Q.7:

Given; $\vec{A} = \hat{i} + \hat{j}$,

$$\vec{B} = \hat{i} + 2\hat{k} \text{ and}$$

$$\vec{C} = 2\hat{j} + \hat{k}$$

a) Find $(\vec{A} \times \vec{B}) \times \vec{C}$ and compare it with $\vec{A} \times (\vec{B} \times \vec{C})$

b) Find $\vec{A} \cdot \vec{B} \times \vec{C}$ and compare it with $\vec{A} \times \vec{B} \cdot \vec{C}$

Solution:

(a)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{i} - 2\hat{j} - \hat{k}$$

Then;

$$(\vec{A} \times \vec{B}) \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -2\hat{j} + 4\hat{k}$$

A similar calculation gives $\vec{A} \times (\vec{B} \times \vec{C}) = 2\hat{i} - 2\hat{j} + 3\hat{k}$. Thus the parentheses that indicate which cross product is to be taken first are essential in the vector triple product.

Given; $\vec{A} = \hat{i} + \hat{j}$,

$$\vec{B} = \hat{i} + 2\hat{k} \text{ and}$$

$$\vec{C} = 2\hat{j} + \hat{k}$$

b) Find $\vec{A} \cdot \vec{B} \times \vec{C}$ and compare it with $\vec{A} \times \vec{B} \cdot \vec{C}$

Solution

$\vec{B} \times \vec{C} = -4\hat{i} - \hat{j} + 2\hat{k}$. Then;

$$\vec{A} \cdot \vec{B} \times \vec{C} = (1)(-4) + (1)(-1) + (0)(2) = -5$$

We have: $\vec{A} \times \vec{B} = 2\hat{i} - 2\hat{j} - \hat{k}$. Then;

$$\vec{A} \times \vec{B} \cdot \vec{C} = (2)(0) + (-2)(2) + (-1)(1) = -5$$

Parentheses are not needed in the scalar triple product since it has meaning only when the cross product is taken first. In general, it can be shown that:

$$\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

As long as the vectors appear in the same cyclic order the result is the same. The scalar triple products out of this cyclic order have a change in sign.

Q.8: Express the unit vector which points from $(z = h)$ on the z-axis toward $(r, \phi, 0)$ in cylindrical coordinates.

Solution:

First, let's represent the two points in Cartesian coordinates to make vector subtraction straight

The point on the z-axis is $\vec{r}_1 = (0, 0, h)$.

The point in the xy-plane is $\vec{r}_2 = (r \cos \phi, r \sin \phi, 0)$.

The vector pointing from \vec{r}_1 toward \vec{r}_2 is $\vec{R} = \vec{r}_2 - \vec{r}_1$.

$$\vec{R} = \vec{r}_2 - \vec{r}_1$$

$$\vec{R} = (r \cos \phi - 0)\hat{x} + (r \sin \phi - 0)\hat{y} + (0 - h)\hat{z} = r \cos \phi \hat{x} + r \sin \phi \hat{y} - h \hat{z}$$

Now we have to change from Cartesian to Cylindrical

$$\rho^2 = x^2 + y^2$$

$$\rho = \sqrt{(r \cos \phi)^2 + (r \sin \phi)^2} = \sqrt{r^2 (\cos^2 \phi + \sin^2 \phi)} = r$$

$$\tan \phi = \frac{y}{x} = \frac{r \sin \phi}{r \cos \phi} = \tan \phi = 0$$

$$z = z \rightarrow z = -h$$

$$\text{So } \vec{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{k}$$

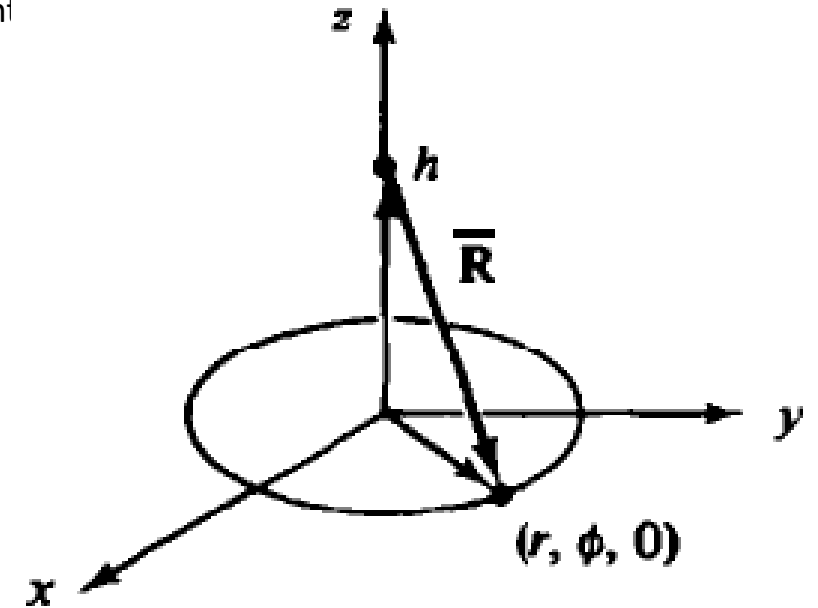
$$\vec{R} = r \hat{\rho} + 0 - h \hat{k} = r \hat{\rho} - h \hat{k}$$

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$R = |\vec{R}| = \sqrt{(r \cos \phi)^2 + (r \sin \phi)^2 + (-h)^2} = \sqrt{r^2 + (\cos \phi + \sin \phi)^2 + h^2}$$

$$|\vec{R}| = \sqrt{r^2 + h^2}$$

$$\text{So, } \hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{r \hat{\rho} - h \hat{k}}{\sqrt{r^2 + h^2}}$$



Q.9: Express the unit vector which is directed toward the origin from an arbitrary point on the plane $z = -5$, as shown in following figure.

Solution:

To find the unit vector, we'll first define the position vector of an arbitrary point on the plane $z=-5$.

Let's call this point P.

The coordinates of point P can be represented as $(x,y,-5)$, where x and y are arbitrary.

Position Vector

The position vector r_P of this point, originating from the origin, is:

$$r_P = x\hat{i} + y\hat{j} + 5\hat{k}$$

Vector Toward the Origin

The vector directed from the point P toward the origin is the negative of the position vector r_P .

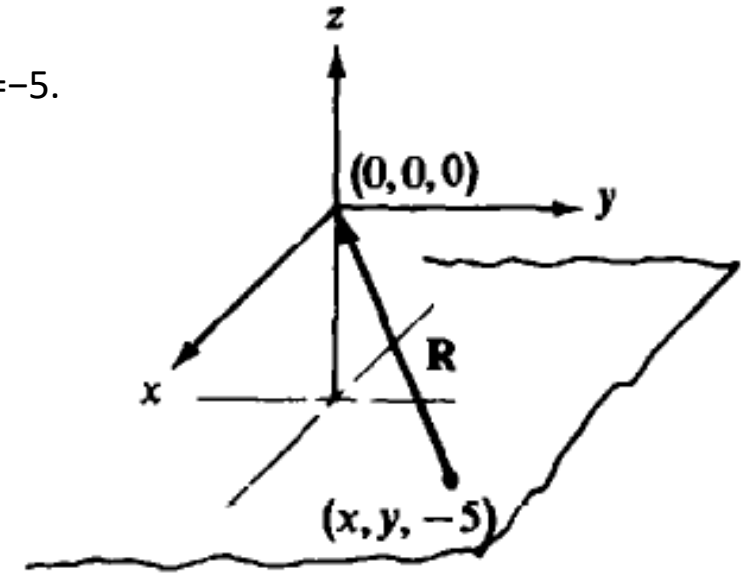
Let's call this vector \vec{R} .

$$\vec{R} = (0 - r_P) = (0 - x)\hat{i} + (0 - y)\hat{j} + (0 + 5)\hat{k} = -x\hat{i} - y\hat{j} + 5\hat{k}$$

Therefore; Unit Vector

A unit vector is a vector with a magnitude of 1. To find the unit vector \hat{R} in the same direction as \vec{R} , we divide the vector \vec{R} by its magnitude $|\vec{R}|$.

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{-x\hat{i} - y\hat{j} + 5\hat{k}}{\sqrt{x^2 + y^2 + 25}}$$



Q.10: Using the appropriate differential elements, show that:

- a) the circumference of a circle of radius ρ_o is $2\pi\rho_o$.
- b) the surface area of a sphere of radius r_o is $4\pi r_o^2$.
- c) the volume of a sphere of radius r_o is $(4/3)\pi r_o^3$.

Solution:

(a) To find the circumference of a circle, we can sum the lengths of small arc segments that make up the circle. We use a polar coordinate system with the differential length element $dl = \rho d\theta$, where ρ is the radius and $d\theta$ is the very small angle.

For a circle of constant radius $\rho = \rho_o$, the circumference L is the integral of the differential length element over a full rotation, from $\theta = 0$ to $\theta = 2\pi$:

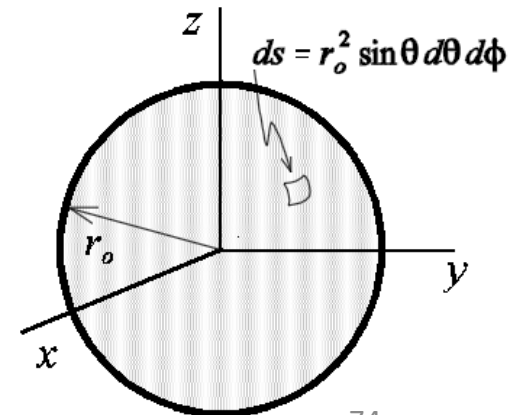
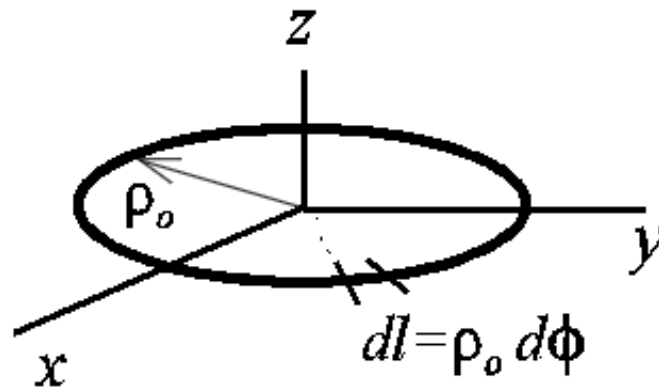
$$L = \int dl = \int_0^{2\pi} \rho_o d\phi = \rho_o \int_0^{2\pi} d\phi = \rho_o [\phi]_0^{2\pi}$$

$$L = 2\pi\rho_o$$

(b) To find the surface area of a sphere, we sum the areas of infinitesimally small patches on its surface. We use a spherical coordinate system with the differential surface area element $dA = (r d\theta)(r \sin\theta d\phi) = r^2 \sin\theta d\theta d\phi$, where r is the radius, θ is the polar angle, and ϕ is the azimuthal angle.

For a sphere of constant radius $r = r_o$, the surface area A is a double integral of the differential area element over the full range of angles: θ from 0 to π and ϕ from 0 to 2π .

$$\begin{aligned} S &= \iint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r_o^2 \sin\theta d\theta d\phi \\ &= r_o^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta d\theta d\phi = r_o^2 \int_{\phi=0}^{2\pi} [-\cos\theta]_0^{\pi} d\phi = r_o^2 (2)(2\pi) \\ S &= 4\pi r_o^2 \end{aligned}$$

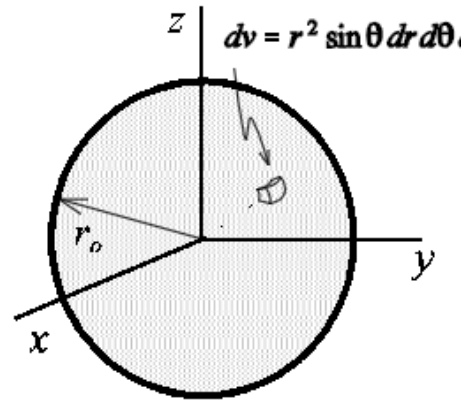


c) Volume of a Sphere of Radius r_0

To find the volume of a sphere, we sum the volumes of infinitesimally small volume elements. We use a spherical coordinate system with the differential volume element $dV=r^2\sin\theta drd\theta d\phi$, where r is the radius, θ is the polar angle, and ϕ is the azimuthal angle.

The volume V is the triple integral of the differential volume element over the entire sphere: the radius r from 0 to r_0 , θ from 0 to π , and ϕ from 0 to 2π .

$$\begin{aligned} V &= \iiint dV \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r_0} r^2 \sin\theta dr d\theta d\phi \\ &= \int_0^{r_0} r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= \left[\frac{r^3}{3} \right]_0^{r_0} [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} \\ &= \left(\frac{r_0^3}{3} \right) (2)(2\pi) \\ V &= \frac{4}{3}\pi r_0^3 \end{aligned}$$



Q.11:

Obtain the expression for the volume of a sphere of a radius a from the differential volume.

Solution:

The expression for the volume of a sphere of radius a can be obtained by integrating the differential volume element in spherical coordinates. The spherical coordinate system is ideal for this problem due to the spherical symmetry of the object.

From the fig.;

The differential volume element, dV , in the spherical coordinate system is given by:

$$dV = r^2 \sin\theta \, dr d\theta d\phi$$

Radial distance (r): The distance from the origin ranges from 0 to the radius of the sphere, so $0 \leq r \leq a$.

Polar angle (θ): The angle from the positive z -axis must sweep from the top pole to the bottom pole, so $0 \leq \theta \leq \pi$.

Azimuthal angle (ϕ): The angle in the xy -plane must complete a full circle, so $0 \leq \phi \leq 2\pi$

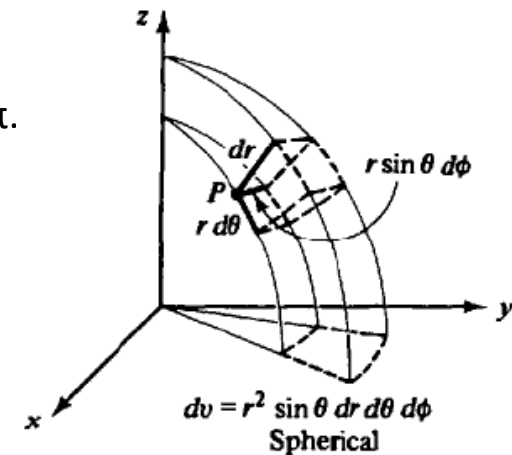
$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^a r^2 \sin\theta \, dr d\theta d\phi$$

1. Integrate with respect to r : $\int_0^a r^2 \sin\theta \, dr = \sin\theta \left[\frac{r^3}{3} \right]_0^a = \sin\theta \left(\frac{a^3}{3} - 0 \right) = \frac{a^3}{3} \sin\theta$

2. Integrate with respect to θ : $\int_0^{\pi} \frac{a^3}{3} \sin\theta \, d\theta = \frac{a^3}{3} [-\cos\theta]_0^{\pi} = \frac{a^3}{3} (-\cos\pi - (-\cos 0)) = \frac{a^3}{3} (-1(-1) + 1) = \frac{a^3}{3} (2) = \frac{2a^3}{3}$

3. Integrate with respect to ϕ : $\int_0^{2\pi} \frac{2a^3}{3} \, d\phi = \frac{2a^3}{3} [\phi]_0^{2\pi} = \frac{2a^3}{3} (2\pi - 0) = \frac{4\pi a^3}{3}$

$$V = \frac{4}{3} \pi a^3 \, \text{m}^3$$



Q.12:

Use the cylindrical coordinate system to find the area of the curved surface of a right circular cylinder, where: $\rho = 2\text{m}$, $h = 5\text{m}$ and $30^\circ \leq \phi \leq 120^\circ$. See the following figure.

Solution:

From the figure;

Radius: $\rho = 2\text{m}$

height: $z = 5\text{m}$

Angle: $\phi = \frac{\pi}{6}$ (30°) to $\frac{2\pi}{3}$ (120°)

The area of the curved surface of a right circular cylinder can be found by integrating the differential area element dA in the cylindrical coordinate system.

The differential area element for the curved surface is given by

$$dA = \rho d\phi dz$$

The area A is given by the double integral of the differential area element over the specified ranges:

$$A = \int_{z=0}^5 \int_{\phi=\pi/6}^{2\pi/3} 2 d\phi dz$$

First, we integrate with respect to ϕ :

$$A = \int_{z=0}^5 \pi dz$$

Next, we integrate with respect to z :

$$A = 5\pi \text{ m}^2$$

