# Foundations of Mathematics

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## Lecture 1: Introduction to the Foundations of Mathematics

### Introduction

Mathematics is the study of numbers, patterns, and structures. It has practical applications in natural sciences, engineering, and various other fields. Mathematics is fundamental in the development of other scientific fields like physics, engineering, and computing.

### Chapter One

- Propositions (Statements)
- Mathematical Proof
- Quantifiers

**Definition 1.1. (**Mathematical Logic**)**

Mathematical logic is the study of reasoning and the principles of valid inference in mathematics. A logical statement is a sentence that can be classified as either true or false.

**Definition 1.2. (**Proposition**)**

A proposition is a declarative statement that is either 'True' (T) or 'False' (F) but cannot be both true and false simultaneously. Letters such as p, q, r, and so forth are used to represent propositions.

**Examples: 1.1.**

- "The number 2 is a prime number" (True)
- "The number 5 is an even number" (False)
- " $x + y = 0$ " (Not a proposition)

**Definition 1.3. (**Negation of a Proposition**)**

Let p be a proposition. The negation of p is called 'not P' and is denoted by (~p).

| p | ~p |
|---|---|
| T | F |
| F | T |

**Examples: 1.2.**

- " P: 3 < 5" , "~p: 3 >= 5"
- "q: 2 + 1 = 5" , " ~~q: 2 + 1 ≠ 5"
- "r: The square has four sides" , "~r: The square has not four sides"

**Remark 1.1.**

If p is a proposition, then $\sim(\sim p) = p$ .

## Compound Propositions:

Propositions are divided into two types:

1. **Primitive proposition:**

   If it cannot be divided into simpler propositions.

2. **Composite proposition:**

   If it is composed of more than one primitive proposition using logic connective operators.

## Logic Connective Operators:

### i. Conjunction Operator (and , ∧)

Let p and q are two propositions. the conjunction of p and q is denoted by "$p \wedge q$" and read "p and q". the table truth of the conjunction of p and q is

| p | q | p∧ $q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Examples: 1.3.**

Find the truth value of the following statements:

a) "2 + 2 = 4 and 2 + 3 = 5"

   $T \wedge T = T$

b) "3 is a prime number and $\pi$ is a rational number "

   $T \wedge F = F$

c) "p: $x + y = y + x, x, y \in R$ and q: 2 > 10"

   $T \wedge F = F$

Properties of conjunction operator (and , ∧)

Let p, q, r be any propositions. Then

1. p∧ q = q ∧ p (commutative)
2. (p ∧ q) ∧ r = p ∧ (q ∧ r) (associative)
3. p ∧ p = p (idempotent law)
4. p ∧ T = p (identity law)
5. p ∧ F = F
6. p ∧ ~p = F

The proof of (1-6) by using the truth table of "and".

## ii. Disjunction Operator (Or ,∨)

Let p and q are two propositions. the disjunction of p and q is denoted by "$p$ ∨q" and read "p or q". the table truth of the conjunction of p and q is

| p | q | p∨q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Examples: 1.4.**

Find the truth value of the following statements:

a) "2 + 2 = 4 or 2 + 3 = 5"
   T ∨T=T
b) "3 is a prime number and $\pi$ is a rational number "
   T ∨F=T
c) "p: $x + y = y + x, x, y \in R$ and q: 2 > 10"
   T ∨F=T

Properties of disjunction operator (or ,∨)

Let p, q, r be any propositions. Then

1. p∨ q = q ∨ p (commutative)
2. (p ∨ q) ∨ r = p ∨ (q ∨ r) (associative)
3. p ∨ p = p (idempotent law)
4. p ∨ T = T (identity law)
5. p ∨ F = p
6. p ∨ ~p = T

The proof of (1-6) by using the truth table of "or".

## Exercises

**Exercise 1:** Wright the truth table of these statements

a) $\sim p \vee q$

b) $\sim \boldsymbol{p} \vee (\boldsymbol{p} \wedge \sim \boldsymbol{q})$

**Exercise 2:** Is the statement "The number 7 is a prime number" true or false? Explain your answer.

**Exercise 3:** Let p, q, r be propositions such that

P: 29 is a prime number , q: $x - x = 0$ , r: $-3 \in N$. Find the truth value of the following statements:

c) $(p \vee q) \vee r$

d) $\sim q \vee r$

e) $\sim(\sim p \vee q)$

f) $(p \wedge q) \vee (p \vee r)$

**Exercise 4:** Evaluate the truth value of the following logical expressions:

1) $((5 > 3)\ and\ (8 = 8))\ or\ ((7 < 2)\ and\ (4 > 1))$
2) $((10 = 10)\ and\ (12 <= 15))\ or\ ((5 >= 6)\ and\ (7 < 7))$
3) $(false\ or\ (true\ and\ false))\ and\ (true\ or\ (false\ and\ true))$
4) $((3 < 9)\ and\ (6 >= 6))\ or\ (not\ (5 = 5))$
5) $(true\ and\ (false\ or\ (true\ and\ false)))\ and\ (true\ or\ false)$

**Exercise 5:**. Determine if the second statement is a logical consequence of the first statement:

1) "If it rains, the ground will be wet." and "The ground is wet." — Is it a logical consequence that "It rained"?
2) "If I study, I will pass the exam." or "If I do not study, I will fail." — Is it a logical consequence that "I will pass if I study"?

3) "All cats are animals." and "Fluffy is a cat." — Is it a logical consequence that "Fluffy is an animal"?

**Exercise 6:**. Construct the truth table for the following expressions and determine whether they are tautologies, contradictions, or contingencies:

a. (p and (q or r)) or (not q and not r)
b. ((p or q) and (not p or r)) and (p or (q and not r))
c. (p or q) and (not (p and q))
d. (p and not q) or (not p and q)

## References


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