Foundations of Mathematics

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Lecture 2.: Conditional Operator

Introduction

The Conditional Operator in mathematical logic is a tool used to connect two statements, establishing a conditional relationship between them. This operator is also known as "implies" or "conditional proposition" and is usually written as:
$p \rightarrow q$
Where:
- p represents the antecedent (the first statement).
- q represents the consequent (the second statement).

The conditional statement "if p is true, then q is also true" is valid in all cases except when p is true and q is false. In that case, the conditional statement is false.

Truth Table for the Conditional Operator:

- If the antecedent is true and the consequent is true, the conditional statement is true.
  - If the antecedent is true but the consequent is false, the conditional statement is false.
  - If the antecedent is false, the conditional statement is true regardless of the consequent.

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Examples: 1.5.

If p: "The number 4 is even" and q: "The number 2 is prime," then the conditional statement $p \rightarrow q$ says "If the number 4 is even, then the number 2 is prime." In this case, the statement is true because both the antecedent and the consequent are true.

Remark 1.2.

1) In the conditional statement "$p \rightarrow q$", P is called the hypothesis and q is called the conclusion. The interpretation of $p \rightarrow q$ is as follows:

a. If p is true, then q is true.
b. q is true if p is true.
c. q is true only if p is true.
d. p is a sufficient condition for q.
e. q is a necessary condition for p.

2) Converse, Inverse, and Contrapositive

The converse of $p \rightarrow q$ is $q \rightarrow p$. The inverse of $p \rightarrow q$ is $\sim P \rightarrow \sim q$, and the contrapositive is $\sim q \rightarrow \sim p$.

Example: 1.5.

Find the truth value of the following statements:
1. If 2 is a prime number, then 2 is an odd number.
Truth value: False

2. If -1 = 1, then $(-1)^2 = (1)^2$.
Truth value: True

Properties of Conditional Operator(if then , →)

Let p, q, and r be propositions. Then, the following properties hold:

1. $(p \rightarrow q) \neq (q \rightarrow p)$
2. $(p \rightarrow q) \rightarrow r \neq p \rightarrow (q \rightarrow r)$

Bi-Conditional Operator

The bi-conditional operator, represented as $p \leftrightarrow q$, is defined as the logical equivalence of p and q. That is, $p \leftrightarrow q$ is true if and only if both p and q are either true or false together.

Truth Table for the Bi-Conditional Operator:

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

Example: 1.6.(a)

Find the truth value of the following bi-conditional statement:

1. $(x > 0) \leftrightarrow (2x > 0)$

If $x > 0$, then $2x > 0$: True
If $x \leq 0$, then $2x \leq 0$: True
Therefore, the truth value is: True

Properties of Bi-Conditional Operator(if and only if , ↔)

Let p, q, and r be propositions. Then, the following properties hold:

1. $(p \leftrightarrow q) = (q \leftrightarrow p)$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$

Example: 1.6.(b)

Wright the truth table of these statements $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \rightarrow \sim \boldsymbol{q})$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \wedge (\sim p \rightarrow \sim q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Exercises

Exercise 1: Find the truth value of the following

$[(\text{If } 2+3=4 \text{ then } x+4=4+x)$ and 8 is an even number$]$ if and only if $[2 \leq -10 \; or \; 2 \geq -10]$

Exercise 2: Let p be a propositions. Then find the truth value of the following

$p \leftrightarrow T$, $p \leftrightarrow F$, $p \leftrightarrow \sim p$, $p \leftrightarrow p$

Exercise 3: Let P: 29 is a prime number , q: $x - x = 0$, r: $-3 \in N$. Find the truth value of the following statements:

a) $(p \leftrightarrow q)$
b) $(p \leftrightarrow r) \vee q$
c) $(p \leftrightarrow q) \vee (q \leftrightarrow \sim r)$

Exercise 4: Evaluate the truth value of the following logical expressions:

1) $if\ ((-4\ >\ 3) then\ (3\ =\ 3))\ or\ (if\ (5\ <\ 2)\ then\ (4\ >\ 1))$
2) $\big((10\ =\ 10) and\ (12\ <=\ 15)\big) if and\ only\ if\ \ ((5\ >=\ 6)\ and\ (7\ <\ 7))$
3) $(false\ \ iff\ \ (true\ and\ false))\ and\ (if\ true\ then\ \ (false\ and\ true))$

Exercise 5:. Construct the truth table for the following statements:

a. ~p ∧ q
b. (p ∧ q) → (p ∨ q)
c. (p ⟶ q) ∨ ~ (q ↔ p)
d) $\boldsymbol{p} \vee (\sim \boldsymbol{q}) \leftrightarrow \sim \boldsymbol{p} \wedge \boldsymbol{q}$
e) $[(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \longrightarrow \boldsymbol{r})] \longrightarrow (\boldsymbol{p} \wedge \sim \boldsymbol{r})$
f) $[\boldsymbol{p} \longrightarrow \boldsymbol{q}] \leftrightarrow (\sim \boldsymbol{q} \longrightarrow \boldsymbol{p} \wedge \boldsymbol{q})$

References

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