Foundations of Mathematics

Instructor: Dr. Uday Jabbar Quaez

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Department of Mathematics, College of Education, Al-Mustansiriya University

Lecture 3.: Logical Equivalence

Introduction

In the study of logic, propositions are statements that can either be true or false. Logical equivalence is an essential concept in propositional logic, which states that two propositions have the same truth value across all possible truth assignments. Understanding logical equivalence is fundamental to simplifying logical expressions and proving their validity.

In this lecture, we will explore the definitions and properties of logical equivalence, including common laws such as De Morgans Laws and the Distributive Laws. We will also practice simplifying logical statements using truth tables and laws of logical equivalence.

Definition of Logical Equivalence:

Two statements (propositions) that have the same truth values are called logically equivalent. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

Examples: 1.7.

Show that $\sim(p \vee q) = \sim p \wedge \sim q$ (logically equivalent).

Hint: Use a truth table.

Solution:

p	q	$\mathbf{p \vee q}$	$\sim(\mathbf{p \vee q})$	$\sim\mathbf{p}$	$\sim q$	$\sim\mathbf{p} \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Definition 1.4. Let p, q, and r be propositions. Then, define the following

1. $(p \wedge q) = \sim(\sim p \vee \sim q)$
2. $(p \rightarrow q) = \sim p \vee q$
3. $(p \leftrightarrow q) = (p \rightarrow q) \wedge (q \rightarrow p)$

De Morgans Laws

Theorem 1.3. (*De Morgans Laws*)

These laws allow the negation of conjunctions and disjunctions to be transformed:

1. $\sim(p \wedge q) = \sim p \vee \sim q$ *(H.W)*
2. $\sim(p \vee q) = \sim p \wedge \sim q$ *(H.W)*

Examples: 1.8. Simplify the following statements using laws of logical equivalence:

1. $\sim(p \wedge \sim q)$
2. $\sim(\sim p \rightarrow q)$
3. $\sim(\sim p \leftrightarrow q)$ *(H.W)*

Solution: Use De Morgans law and implications definition for step-by-step simplification

1.

$$\sim(p \wedge \sim q) = \sim p \vee \sim\sim q \qquad (De\ Morgans\ Laws)$$

$$\sim(p \wedge \sim q) = \sim p \vee q \qquad (since\ \ \sim\sim q = q)$$

$$\sim(p \wedge \sim q) = p \rightarrow q \qquad (def. \rightarrow)$$

2.

$$\sim(\sim p \rightarrow q) = \sim(\sim\sim p \vee q) \qquad (def. \rightarrow)$$

$$= \sim(p \vee q) \qquad (\sim\sim p = p)$$

$$= \sim p \wedge \sim q \qquad (De\ Morgans\ Laws)$$

Important Laws of Logical Equivalence:

1. ***Commutative Law:***

$$p \wedge q = q \wedge p$$
$$p \vee q = q \vee p$$

$$p \leftrightarrow q = q \leftrightarrow p$$

2. ***Associative Law:***

$$(p \wedge q) \wedge r = p \wedge (q \wedge r)$$

$$(p \vee q) \vee r = p \vee (q \vee r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

3. ***Distributive Law: (left)***

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$
$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$
$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$
$$p \vee (q \rightarrow r) = (p \vee q) \rightarrow (p \vee r)$$
$$p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$$

(Right) in the same equivalents on the side right**.**

4. ***Idempotent Law:***

$$p \wedge p = p$$
$$p \vee p = p$$

5. ***Identity Law:***

$$p \wedge T = p$$
$$p \vee F = p$$

Examples: 1.9. Simplify the following statements using laws of logical equivalence:

1. $\sim p \vee (p \wedge \sim q)$
2. $\sim(\sim p \rightarrow q) \vee q$ *(H.W)*
3. $p \vee (\sim p \leftrightarrow q)$ *(H.W)*

Solution:

1.

$$\sim p \vee (p \wedge \sim q) = (\sim p \vee p) \wedge (\sim p \vee \sim q) \qquad (Distributive\ Law)$$
$$= (T) \wedge (\sim p \vee \sim q) \qquad (since \sim p \vee p = T)$$

$$= (\sim p \vee \sim q) \qquad (Identity\ Law)$$

Examples: 1.10. Prove that the following statement:

$$\sim(p \vee (\sim p \wedge q) = \sim p \wedge \sim q$$

Solution:

$$\begin{aligned}
\sim(p \vee (\sim p \wedge q) &= \sim p \wedge \sim(\sim p \wedge q) && (De\ Morgans\ Laws)\\
&= \sim p \wedge (\sim\sim p \vee \sim q) && (De\ Morgans\ Laws)\\
&= \sim p \wedge (p \vee \sim q) && (\ since \sim\sim p = p\ )\\
&= (\sim p \wedge p\ ) \vee (\sim p \wedge \sim q) && (Distributive\ Law)\\
&= (T\ ) \vee (\sim p \wedge \sim q) && (since \sim p \vee p = T)\\
&= (\sim p \wedge \sim q) && (Identity\ Law)
\end{aligned}$$

Conclusion:

Logical equivalence is a crucial tool in propositional logic, aiding in the simplification and manipulation of logical expressions. By mastering the laws of logic, such as De Morgan's and the Distributive Laws, we can simplify complex expressions and better understand logical arguments.

Exercises

Exercise 1: prove that the following properties:

1. $p \rightarrow \mathrm{p} = \mathrm{T}$
2. $\sim p \rightarrow \mathrm{p} = \mathrm{p}$
3. $p \rightarrow \mathrm{T} = \mathrm{T}$
4. $T \rightarrow \mathrm{p} = \mathrm{T}$
5. $p \rightarrow \mathrm{F} = \sim p$
6. $F \rightarrow \mathrm{p} = T$
7. $p \rightarrow \mathrm{q} = \sim q \rightarrow \sim \mathrm{p}$
8. $p \rightarrow \mathrm{q} = (p \wedge \sim q) \rightarrow \sim p$
9. $p \rightarrow \mathrm{q} = (p \wedge \sim q) \rightarrow (r \wedge \sim r)$
10. $\sim(p \rightarrow \mathrm{q}) = p \wedge \sim q$

Exercise 2: Let p be a propositions. Then find the truth value of the following

1. $p \leftrightarrow \mathrm{p} = \mathrm{T}$
2. $\sim p \leftrightarrow \mathrm{p} = \mathrm{F}$

3. $p \leftrightarrow \mathrm{q} = q \leftrightarrow \mathrm{p}$
4. $p \leftrightarrow \mathrm{q} = \sim p \leftrightarrow \sim q$
5. $\sim p \leftrightarrow \mathrm{q} = p \leftrightarrow \sim q$
6. $\sim(p \leftrightarrow \mathrm{q}) = \sim p \leftrightarrow q$
7. $\sim(p \leftrightarrow \mathrm{q}) = p \leftrightarrow \sim q$

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