Foundations of Mathematics

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Lecture 4.: Mathematical Proof and Methods of Proving Statements

Introduction

Mathematical proof is the foundation of all mathematical reasoning. It is a formal argument that establishes the truth of a mathematical statement using logical steps. In this lecture, we will explore different methods of proving mathematical statements, including direct proof and proof by contradiction. We will also examine conditional statements and introduce the concept of open sentences and truth sets, all of which are crucial for understanding how mathematical assertions are validated..

Methods of Proving Mathematical Statements (or Theorems):

1. Direct Proof of a conditional statement p → q

A direct proof is a logical process that starts from the assumption of the hypothesis and works step by step to reach the conclusion.

Definition 1.5. An integer x is called even if there exists k such that $x = 2k$.

Definition 1.6. An integer x is called odd if there exists k such that $x = 2k + 1$.

Theorem 1.4. If x is an odd natural number, then x^2 is odd.

Proof: Assume that x is an odd natural number. We must prove x^2 is odd.

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$$x^2 = x.x = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

Theorem 1.5. (Homework): If x is an even natural number, then x^2 is even.

Theorem 1.6. (Homework): The sum of two even natural numbers is even.

2. Direct Proof of a conditional statement $p \leftrightarrow q$

To prove a proposition in the form $p \leftrightarrow q$, we prove its equivalence. i.e.,

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

Theorem 1.7. x is odd number if and only if $x + 1$ is an even number.

Proof: Let p: x is odd number and q: x+1 is an even number

To prove $p \rightarrow q$ and $q \rightarrow p$

Prove p → q

Let $x \in O$ then $x = 2k + 1, k \in Z$

$x + 1 = 2k + 2 = 2(2k + 1)$, $k + 1 \in Z$

Assume that $r = k + 1$ then $x + 1 = 2r$;, $r \in Z$

$$x + 1 \in E$$

Prove q → p:

Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k, k \in Z$

$x = 2k - 1$ ... (1)

Since $k \in Z$, then $r = k - 1 \in Z$

$$k = r + 1 \quad \text{... (2)}$$

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1 \in O$

Theorem 1.8. (Homework): x is an even number if and only if x^2 is even number.

Theorem 1.9.(a) (Homework): x is an odd number if and only if x^2 is odd number.

Theorem 1.9.(b) Prove that if $x \neq 0$, then $x^{-1} \neq 0$

Proof Let $\boldsymbol{p: x \neq 0, q: x^{-1} \neq 0}$

we have to prove $\mathbf{p \rightarrow q}$ is true statement , assume $\sim (\mathbf{p} \rightarrow \mathbf{q})$ is true statement since $\sim(p \rightarrow q) = p \wedge \sim q$, then $p \wedge \sim q$ is true statement

$\boldsymbol{x^{-1}}.x = 0$ *but* $\boldsymbol{x^{-1}}.x = 1$ *this impels* 1= 0 which is a contradiction C!

So $\sim(\mathbf{p} \rightarrow \mathbf{q})$ is false statement .

Therefore, $\mathbf{p} \rightarrow \mathbf{q}$ is true statement.

Proof by Contradiction

Proof by contradiction involves assuming the opposite of what you want to prove, and then showing that this assumption leads to a contradiction.

Theorem 1.10. prove that if x is an odd number, then x^2 is odd.

Proof: Assume that x^2 is odd number. To prove x is an odd number by contradiction. Assume that $x \in E$

Then $x = 2k$, $k \in Z$

$x^2 = 4k^2 = 2 * 2k^2$, $2k^2 \in Z$

$x^2 \in E$ this is contradiction with assumption. This implies x is an odd number.

Theorem 1.11. Prove that: If $n = ab$ where a and b are positive integers, then a $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer (**hypothesis)**

$q: a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **(conclusion)**

Assume that the conclusion is false this means $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$) is true.

But $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n}) = \sim(a \leq \sqrt{n}) \wedge \sim(b \leq \sqrt{n})$ [De Morgan's law]

$$= (a > \sqrt{n}) \wedge (b > \sqrt{n})$$

Therefore, $ab > n$ this implies to contradiction with the hypothesis.

Thus, $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is true.

Proof by mathematical induction

The principle of mathematical induction is one of the methods of mathematical proof. By using it, we can prove the validity of a statement $p(n)$for every natural number n, starting with verifying it for a base case and then proving that if it holds for $n = k$, it also holds for $n = k + 1$.

This process involves the following steps:

1. Check that $p(n)$is true for n=1 .

2. Assume that $p(n)$ is true for $n = k$ (where $k > 1$).

3. Prove that $p(n)$ is true for $n = k + 1$.

If the statement holds for n=1 and if $p(n)$ being true for $n = k$ implies it is also true for $n = k + 1$, then by induction, $p(n)$ is true for all $n \geq 1$.

Examples: 1.11. prove that the following statement is true by mathematical induction.

1) $1 + 2 + \cdots + \mathrm{n} = \frac{\mathrm{n}(\mathrm{n}+1)}{2}$

2) $1^2 + 2^2 + \cdots + \mathrm{n}^2 = \frac{\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)}{6}$

3) $1^3 + 2^3 + \cdots + \mathrm{n}^3 = \left(\frac{\mathrm{n}(\mathrm{n}+1)}{2}\right)^2$

Solution 1)

To prove the statement is true we use the mathematical induction

If n=1 then $1=\frac{1\times(1+1)}{2}$ is true

Assume that the statement is true when $n = k + 1$, to prove the statement is true when $n = k + 1$

$$1 + 2 + \cdots + \mathrm{k} + \mathrm{k} + 1 = \frac{\mathrm{k}(\mathrm{k}+1)}{2} + k + 1 = \frac{\mathrm{k}(\mathrm{k}+1)+2k+2}{2} = \frac{\mathrm{k}^2+3k+2}{2} = \frac{(\mathrm{k}+1)(\mathrm{k}+2)}{2}$$

Therefore, the statement is true for all values of n.

Definition 1.7. (Variable) An alphabetic letter $x, y, z, \ldots$ which represents a number that is either arbitrary or unknown.

Definition 1.8. (Open Sentence) A sentence is called open sentence (or propositional function), if it contains one or more variables. Open sentence is denoted by p(x), q(x), g(x)…etc.

Examples: 1.12. The following are open sentences:

$p(x)$: x is an odd number.

$q(x, y) = x + y = 5 , x, y \in N$

Examples: 1.13. Let the open sentence $p(x), x > 5$. What are the truth value of $p(7)$ *and* $p(0)$? Which values $x \in N$ that make $p(x)$ true?

Solution: $p(7)$: 7 > 5 is a true proposition

$p(0)$: 0 > 5 is a false proposition

$p(x)$ is a true statement for $x \in \{6,7,8, \dots\}$.

Examples: 1.14. Let the open sentence $q(x, y, z)\colon x + y = z,\ x, y, z \in Z$. What are the truth values of $q(1,1,2)$ *and* $q(-1,1,5)$?

Solution: $q(1,1,2)\colon 1 + 1 = 2$ is a true proposition

$q(-1,1,5)\colon -1 + 1 = 5$ is a false proposition

Definition 1.9. (Solution Set or Truth Set) Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. $T_p = \{x \in A\colon p(x) \text{ is true}\}$

Examples: 1.15. Find the solution set for each of the following open sentences:

1. $p(x), x + 2 > 7$ *and* $A = N$
2. $q(x), x + 2 = 0$ *and* $A = N$
3. $r(x), x + 5 > 1$ *and* $A = N$

Solution: 1. $T_p = \{6,7,8, \dots\}$

2. $T_p = \emptyset$

3. $T_p = N$

Exercises

Exercise 1: Assume we have the following statement: "$x \leq -3$ *or* $x \geq 6$". Which values of x $\in N$ that make the statement true? Which values of x that make the statement false?

Exercise 2: Assume we have the following statement: "$x > 2$ *and* $x < 5$". Which values of $x \in N$ that make the statement true? Which values of x that make the statement false? Discuss all the possible cases.

Exercise 3: Find the following solution sets. Also determine $p(x)$ and A for each solution set

1. $T_p = \{x \in N, -2 < x < 2\}$

2. $T_p = \{x \in Z, -1 < x < 1\}$

References

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