

Foundations of Mathematics

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Lecture 5.: Quantifiers

Introduction

Quantifiers are special symbols used in logic and mathematics to express the quantity of elements in a given set that satisfy a certain property. They transform open sentences into statements that are either true or false.

There are two types of quantifiers:

1. Universal Quantifiers (denoted by " \forall "): Denotes that a property holds for all elements in a set.
2. Existential Quantifiers (denoted by " \exists "): Denotes that a property holds for at least one element in a set...

Universal Quantifiers:

The universal quantifier applies to all elements within a set. The expression " $\forall x \in A, p(x)$ " translates to "for every x in the set A , the proposition $p(x)$ is true."

Examples: 1.15. " $\forall x \in N, x > 0$ "

Translation: For all natural numbers x , x is greater than zero.

This statement is true since every natural number is greater than zero.

Remark 1.12. The universal quantifier $p(x)$ on a domain A is true if and only if $T_p = A$.

Examples: 1.16. Find the truth value of the following open sentences:

1. " $\forall x \in R, x + 1 > x$ "

Let $A = R$ and $p(x): x + 1 > x$

Because $p(x)$ is true for all $x \in R$, the solution set $T_p = R$

\Rightarrow the quantification $\forall x \in R, x + 1 > x$ is **true**.

$$2. \forall x \in N, x < 2$$

Let $A = N$ and $p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3$, $p(3)$ is false.

$$\Rightarrow T_p \neq N$$

$$3. \forall x \in N, (x > 0 \text{ and } x = 0)$$

The statement is **false**, there exists $x = 4 \in N$ such that $4 > 0$ and $4 \neq 0$.

Existential Quantifiers:

The existential quantifier applies when there is at least one element in a set that satisfies the property. The expression "there exists x in A , $p(x)$ " means "there exists an x in set A such that $p(x)$ is true."

Examples: 1.17. $\exists x \in N, x < 0$

Translation: There exists a natural number x that is less than zero.

This statement is false because no natural number is less than zero.

Remark 1.13. The existential quantifier $p(x)$ on a domain A is **true** if and only if $T_p \neq \emptyset$.

Examples: 1.18. Find the truth value of the following open sentences:

$$1. \exists x \in R, x^2 = x$$

$$A = R \text{ and } p(x): x^2 = x$$

$$T_p = \{0, 1\} \Rightarrow \text{the existential quantifier } \exists x \in R, x^2 = x \text{ is true}$$

$$2. \exists x \in N, 3x + 5 = 1$$

$$x = \frac{-4}{3} \notin N \Rightarrow T_p = \emptyset \Rightarrow \exists x \in N, 3x + 5 = 1 \text{ is false}$$

$$3. \exists x \in Z, (x + 1)^2 = 0 \text{ and } x^2 - 1 = 0$$

$$A = Z \text{ and } p(x): (x + 1)^2 = 0 \text{ and } x^2 - 1 = 0$$

$$x = -1 \text{ and } x = \bar{1}$$

$$T_p = \{-1\} \subset Z$$

$$\exists x \in Z, (x + 1)^2 = 0 \text{ and } x^2 - 1 = 0 \text{ is true}$$

De Morgan's Law for the Existential Quantifier:

De Morgan Laws also apply to quantifiers, providing ways to transform statements involving " \forall " and " \exists "

- The negation of a universal quantifier: The negation of " \forall " is " \exists " with the negation of

the property. $\sim [\forall x \in A, \sim p(x)] = \forall x \in A, p(x)$

- The negation of an existential quantifier: The negation of " \exists " is " \forall " with the negation

of the property.

Theorem 1.14. Let $p(x)$ be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A, p(x)] = \exists x \in A, \sim p(x)$
2. $\sim[\forall x \in A, \sim p(x)] = \exists x \in A, p(x)$
3. $\sim[\exists x \in A, p(x)] = \forall x \in A, \sim p(x)$ (**H.W.**)

Proof: 1. $\sim[\forall x \in A, p(x)] = \sim[\sim [\exists x \in A, \sim p(x)]]$ (De Morgan law)

$$= \sim\sim[\exists x \in A, \sim p(x)]$$

$$= \exists x \in A, \sim p(x) \quad [\sim\sim p = p]$$

2. $\sim[\forall x \in A, \sim p(x)] = \sim[\sim [\exists x \in A, \sim\sim p(x)]]$ (De Morgan law)

$$= \exists x \in A, p(x) \quad [\sim\sim p = p]$$

3. $\sim[\exists x \in A, p(x)] = \sim[\exists x \in A, \sim\sim p(x)]$ since $[\sim\sim p = p]$

$$= [\forall x \in A, \sim p(x)] \text{ (De Morgan law)}$$

Nested Quantifiers:

Nested quantifiers occur when two or more quantifiers are used in a single statement, each applying to different sets. For example:

" $\forall x \in A, \forall y \in B, p(x, y)$ "

Meaning: For every x in A and every y in B , the property $p(x, y)$ holds true.

Examples: 1.19.

1. $\forall x \in R, \forall y \in N, x^2 + y^2 \geq 0$ (**True**)

- 2. $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, x + y < 0$ **(False)**
- 3. $\exists x \in \mathbb{R}, \forall y \in \mathbb{N}, x + y = 0$ **(False)**

Theorem 1.15. Let x and y are two variables defined on the sets A and B , respectively and $p(x, y)$ an open sentence. Then:

- 1. $\sim[\forall x \in A, \forall y \in B, p(x, y)] = \exists x \in A, \exists y \in B, \sim p(x, y)$ **(H. W.)**
- 2. $\sim[\exists x \in A, \exists y \in B, p(x, y)] = \forall x \in A, \forall y \in B, \sim p(x, y)$
- 3. $\sim[\forall x \in A, \exists y \in B, p(x, y)] = \exists x \in A, \forall y \in B, \sim p(x, y)$ **(H. W.)**
- 4. $\sim[\exists x \in A, \forall y \in B, p(x, y)] = \forall x \in A, \exists y \in B, \sim p(x, y)$

Proof:

2. Take the L. H. S

$$\begin{aligned} \sim[\exists x \in A, \exists y \in B, p(x, y)] &= \forall x \in A, \sim[\exists y \in B, p(x, y)] \\ &= \forall x \in A, \forall y \in B, \sim p(x, y) \\ &= \text{R. H. S} \end{aligned}$$

Examples: 1.20.

Find the truth values of the following statements and their negations:

- 1. $\forall x \in \mathbb{R} (x \neq 0), \exists y \in \mathbb{R}, xy = 1$

The statement is true because $\forall x \in \mathbb{R} (x \neq 0), \exists y \in \mathbb{R}, y = \frac{1}{x}, x \cdot \frac{1}{x} = 1$

Negation:

$$\sim[\forall x \in \mathbb{R} (x \neq 0), \exists y \in \mathbb{R}, xy = 1] = \exists x \in \mathbb{R} (x \neq 0), \forall y \in \mathbb{R}, xy \neq 1$$

The statement is false, Let $x=2$ and $y = \frac{1}{2}$ then $xy = 1$.

- 2. $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0$ is true

Negation:

$$\sim[\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0] = \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 < 0 \text{ is false}$$

- 3. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, x + y \in \mathbb{N}$ **(H. W.)**
- 4. $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}, x + y \in \mathbb{N}$ **(H. W.)**

2. Find the negation of the following sentences:

Exercises

Exercise 1: Express the following using connective operators and/or quantifiers

1. There exists p , and there exist q such that $pq = 32$.
2. For each x , there exists y such that $x < y$.
3. Each even number is not odd number.
4. For each x , if x is natural number then x is an integer number.
5. For each natural number x , x is even number or x is odd number.

Exercise 2: Find the negation of the following sentences:

1. $\forall x, \forall y, \exists z, x + y + z = 18$
2. There exists y such for each $x, xy \leq 2$
3. $\exists x, [p(x) \rightarrow Q(x)]$

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