

Foundations of Mathematics

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Lecture 7.: Complement of a Set and Difference of Sets

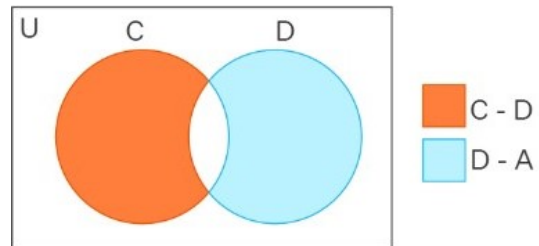
Introduction

In mathematics, the complement and difference of sets are fundamental operations in set theory. These concepts allow us to analyze relationships between sets and their elements. Understanding these operations helps in various mathematical applications, including logic, probability, and computer science.

Definition of Difference of Sets:

For two sets A and B, the difference A and B denoted by $A - B$ is a set of all elements that belong to A but not to B.

Formally: $A - B = \{x \in A \wedge x \notin B\}$



Examples: 2.10. Let $A = N$ and $B = \{1, 2, 3\}$. Then

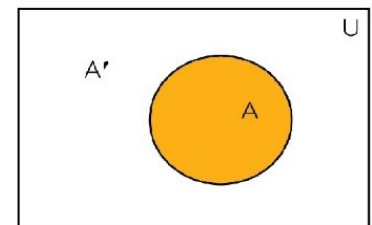
$$A - B = \{4, 5, 6, \dots\}$$

$$A - \emptyset = A$$

$$A - A = \emptyset$$

Definition 2.7. The complement of a set A, within a universal set X, includes all the elements in X that are not in A. It is denoted by A^c or $X - A$ and is defined as:

$$A^c = \{x \in X \wedge x \notin A\}$$



Examples: 2.11. 1) Let $X = N$ and $A = \{1, 2, 3\}$. Then

$$A^c = \{4, 5, 6, \dots\}$$

2) Let $X = N$ and $A = N$. Then $A^c = \emptyset$

3) Let $X = R$ and $A = \emptyset$. Then $A^c = R$

4) Let $X = N, A = \{1, 2\}, B = \{2, 3, 4\}$. Then

$$A^c = N - A = \{3, 4, 5, \dots\}$$

$$B^c = N - B = \{1, 5, 6, 7, \dots\}$$

$$(A \cap B)^c = N - (A \cap B) = N - \{2\} = \{1, 3, 4, 5, \dots\}$$

$$(A \cup B)^c = N - (A \cup B) = N - \{1, 2, 3, 4\} = \{5, 6, 7, \dots\}$$

$$A^c \cap B^c = \{5, 6, 7, \dots\}$$

$$A^c \cup B^c = \{1, 3, 4, 5, \dots\}$$

Theorem 2.7. Let A, B and C be any sets in universal set X . Then

1. $\emptyset^c = X$
2. $A \cap A^c = \emptyset$
3. $A \cup A^c = X$
4. $(A^c)^c = A$
5. $(A \cup B)^c = A^c \cap B^c$
6. $(A \cap B)^c = A^c \cup B^c$
7. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$
8. $A \subseteq B \Leftrightarrow A \cap B^c = \emptyset$

Proof: 1. Assume that $\emptyset^c \neq X, \exists x, x \in X \wedge x \notin \emptyset^c$

$x \in X \wedge x \in \emptyset$ this is C!

$\therefore \emptyset^c = X$

2. Assume that $A \cap A^c \neq \emptyset$

$\exists x \in X, x \in A \cap A^c \Rightarrow x \in A \wedge x \in A^c$ (Def. of \cap)

$\Rightarrow x \in A \wedge x \notin A$ (Def. of Complement set)

this is C! $\Rightarrow \therefore A \cap A^c = \emptyset$

3. Assume that $A \cup A^c \neq X$

$\exists x \in X \wedge x \notin A \cup A^c$

$x \notin A \cup A^c \Rightarrow x \notin A \wedge x \notin A^c$ (*Def. of \cup and De – morgan Lows*)

This is contradiction with the definition of complement set.

$\therefore A \cup A^c = X$

4. $(A^c)^c = A$ to prove $A \subseteq (A^c)^c$ and $(A^c)^c \subseteq A$

Let $x \in A \Rightarrow x \notin A^c$ (*Def. of Complement set*)

$\Rightarrow x \in (A^c)^c \Rightarrow \therefore A \subseteq (A^c)^c$... (1)

Let $x \in (A^c)^c \Rightarrow x \notin A^c$ (*Def. of Complement set*)

$\Rightarrow x \in A \Rightarrow \therefore (A^c)^c \subseteq A$... (2)

From (1) and (2), we have $(A^c)^c = A$

5. $(A \cup B)^c = A^c \cap B^c$ to prove $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in (A \cup B)^c \Rightarrow x \notin A \cup B$ (*Def. of Complement set*)

$\Rightarrow x \notin A \wedge x \notin B$ (*Def of \cup*)

$\Rightarrow x \in A^c \wedge x \in B^c$ (*Def. of Complement set*)

$\Rightarrow x \in A^c \cap B^c$ (*Def. of \cap*)

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$... (1)

Similarity, we prove $A^c \cap B^c \subseteq (A \cup B)^c$... (2)

From (1) and (2), we have $(A \cup B)^c = A^c \cap B^c$

$$6. (A \cap B)^c = A^c \cup B^c$$

to prove $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$

Let $x \in (A \cap B)^c \Rightarrow x \notin A \cap B$ (*Def. of Complement set*)

$\Rightarrow x \notin A \vee x \notin B$ (*Def. of \cap and De – morgan Laws*)

$\Rightarrow x \in A^c \vee x \in B^c$ (*Def. of Complement set*)

$\Rightarrow x \in A^c \cup B^c$ (*Def. of \cup*)

$\therefore (A \cap B)^c \subseteq A^c \cup B^c \dots (1)$

Similarity, we prove $A^c \cup B^c \subseteq (A \cap B)^c \dots (2)$

From (1) and (2), we have $(A \cap B)^c = A^c \cup B^c$

$$7. A \subseteq B \Leftrightarrow B^c \subseteq A^c \quad (\Rightarrow)$$

Suppose that $A \subseteq B$, **to prove** $B^c \subseteq A^c$

Let $x \in B^c \Rightarrow x \notin B$ (*Def. of Complement set*)

Since $A \subseteq B \Rightarrow x \notin A$ (*Def. of subset*)

$\therefore x \in A^c$ (*Def. of Complement set*)

$\therefore B^c \subseteq A^c$

(\Leftarrow) Suppose that $B^c \subseteq A^c$, **to prove** $A \subseteq B$

Let $x \in A \Rightarrow x \notin A^c$ (*Def. of Complement set*)

Since $B^c \subseteq A^c \Rightarrow x \notin B^c$ (*Def. of subset*)

$\therefore x \in B$ (*Def. of Complement set*)

$\therefore A \subseteq B$

Theorem 2.8. Let A, B and C be any sets in universal set X . Then

1. $A - A = \emptyset$ and $A - X = \emptyset$
2. $A - \emptyset = A$ and $\emptyset - A = \emptyset$
3. $A - B \subseteq A$
4. $A \subseteq B \Leftrightarrow A - B = \emptyset$
5. $(A - B) \cap B = \emptyset$
6. $A \cap B = \emptyset \Leftrightarrow (A - B = A) \wedge (B - A = B)$
7. $A - (B \cup C) = (A - B) \cap (A - C)$
8. $A - (B \cap C) = (A - B) \cup (A - C)$
9. $A - A^c = A$ and $A^c - A = A^c$

Proof: 1. To prove $A - A = \emptyset$

Suppose that $A - A \neq \emptyset \Rightarrow \exists x \in X, x \in A - A$

$\Rightarrow x \in A \wedge x \notin A$ (Def. of difference set)

This implies to contradiction. $\therefore A - A = \emptyset$

3. To prove $A - B \subseteq A$

Let $x \in A - B \Rightarrow x \in A \wedge x \notin B$ (Def. of difference set)

$\therefore x \in A$ ($p \wedge T = p$)

4. To prove $A \subseteq B \Leftrightarrow A - B = \emptyset$

(\Rightarrow) Suppose that $A \subseteq B$ to prove $A - B = \emptyset$

Assume that $A - B \neq \emptyset \Rightarrow \exists x \in X, x \in A - B$

$\Rightarrow x \in A \wedge x \notin B$ (Def. of difference set)

But, $A \subseteq B \Rightarrow x \in A \wedge x \notin A$. This implies to contradiction. $\therefore A - B = \emptyset$

(\Leftarrow) Suppose that $A - B = \emptyset$, to prove $A \subseteq B$

Assume that $A \not\subseteq B \Rightarrow \exists x \in X, x \in A \wedge x \notin B$ (Def. of subset)

$\therefore x \in A - B$ (Def. of difference set)

$\Rightarrow A - B \neq \emptyset$ This implies to contradiction. $\therefore A \subseteq B$

5. Assume that $(A - B) \cap B \neq \emptyset \Rightarrow \exists x, x \in (A - B) \cap B$

$\Rightarrow x \in A - B \wedge x \in B$ (Def. of \cap)

$\Rightarrow (x \in A \wedge x \notin B) \wedge x \in B$ (Def. of difference set)

$\Rightarrow x \in A \wedge (x \notin B \wedge x \in B)$ (\wedge is associative)

$\Rightarrow x \in A \wedge F$ this implies that the proposition is false ($p \wedge F = F$)

$\Rightarrow \therefore (A - B) \cap B = \emptyset$

6. (\Rightarrow) Let $A \cap B = \emptyset$, **to prove** $A - B = A \wedge B - A = B$

Let $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$ (Def. of difference set)

$\Leftrightarrow x \in A \wedge T$ (since $A \cap B = \emptyset$)

$\Leftrightarrow x \in A$ ($p \wedge T = p$)

$\therefore A - B = A$

In the Similar method, we prove that $B - A = B$ Type equation here.

(\Leftarrow) Let $A - B = A \wedge B - A = B$, **to prove** $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset \Rightarrow \exists x \in X, x \in A \cap B \Rightarrow x \in A \wedge x \in B$ (Def. of \cap)

$\Rightarrow \exists x, x \in A - B \wedge x \in B$ ($A - B = A$)

$\Rightarrow \exists x, x \in A - B \wedge x \in B$

$\Rightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B$

$\Rightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B)$ (\wedge is associative)

$\Rightarrow \exists x, x \in A \wedge F = F$ ($p \wedge F = F$)

$\therefore A \cap B = \emptyset$

8. To prove $A - (B \cap C) = (A - B) \cup (A - C)$

Let $x \in A - (B \cap C) \Leftrightarrow x \in A \wedge x \notin (B \cap C)$ *(Def. of difference set)*

$\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C)$ *(Def. of \cap and De - morgan Laws)*

$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$ *(\wedge distributives on \vee)*

$\Leftrightarrow x \in (A - B) \vee x \in (A - C)$ *(Def. of difference set)*

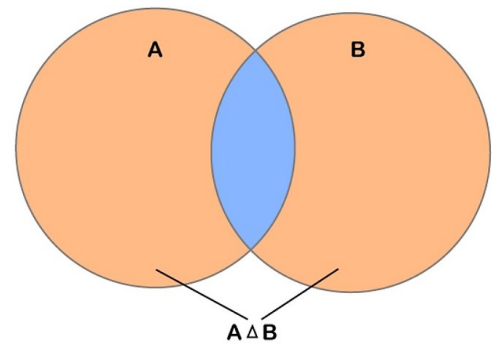
$\Leftrightarrow x \in (A - B) \cup (A - C)$ *(Def of \cup)*

Definition of Symmetric Difference Sets:

The symmetric difference between two sets A and B is denoted by $A \Delta B$ and is defined as follows:

$$A \Delta B = (A - B) \cup (B - A)$$

$$= (A \cup B) - (A \cap B)$$



Examples: 2.12.

Let $A = N$ and $B = \{1, 2, 3\}$. Then

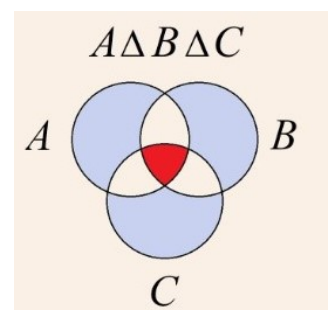
$$A \Delta B = (A \cup B) - (A \cap B) = N - \{1, 2, 3\} = \{4, 5, 6, \dots\}$$

$$A \Delta \emptyset = (A \cup \emptyset) - (A \cap \emptyset) = A$$

$$A \Delta A = \emptyset$$

Theorem 2.9. Let A, B and C be any sets in universal set X . Then

1. $A \Delta A = \emptyset$
2. $A \Delta B = B \Delta A$
3. $A \Delta B = \emptyset \Leftrightarrow A = B$
4. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$



$$5. A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$$

Exercises

Exercise 1: Prove or disprove the following

$$1. A \cup B = A \cap B \Leftrightarrow A = B$$

$$2. A - (B \cup C) = \emptyset \Rightarrow A \subseteq (B \cup C)$$

Exercise 2: Prove or disprove the following

$$1. P(A \cap B) \subseteq P(A) \cap P(B)$$

$$2. (A - B)^c = B \cup A^c$$

Exercise 3: Let $A = \{x \in \mathbb{N} : 1 \leq x < 5\}$, $B = \{2, 4, 6\}$ and $C = \mathbb{N}$.

Find $A - (B \cup C)$, $A - (B \cap C)$

Exercise 4: If $A = \emptyset$ then find $P(A)$ and $P(A^c \cap A)$

Exercise 5: Let $A \cap B = \emptyset$. Is $A - B = B - A$, Explain that?

Exercise 6: Let $X = \{A_n; A_n = \left(\frac{-1}{n}, \frac{1}{n}\right); n \in \mathbb{N}\}$. Find $\bigcap_{n=1}^4 (A_1 - A_n)$, $\bigcup_{n=1}^4 (A_1 - A_n)$

Exercise 7: Let $A = \{x \in \mathbb{E} : -8 \leq x < 9\}$ and $B = \{1, 2, 4, 6\}$ Find $A \Delta B$.

References

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