

Foundations of Mathematics

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Chapter Three

Relations

Lecture 8.: Cartesian Product

Introduction

The Cartesian product is a fundamental concept used to build many ideas in mathematics, engineering, and programming. It is defined as the operation that produces a set of ordered pairs, where the first element of each pair comes from the first set, and the second element comes from the second set.

Ordered pair:

Ordered pair is a mathematical concept used to represent two elements arranged in a specific order where the order of elements is important. It is written in the form (a, b) where a is the first element and b is the second element.

Properties of Ordered Pairs:

Let a, b, c and d be elements. Then:

1. $(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$
2. $(a, b) = (b, a)$ if $a = b$

Cartesian Product:

If A and B are two sets, their Cartesian product is denoted by $A \times B$ and is defined as :

$$A \times B = \{(a, b); a \in A \wedge b \in B\}$$

The size of Cartesian product is calculated as: if A has n elements and B has m elements then $A \times B$ has $m \times n$ elements.

Examples: 3.1. let $A = \{1,2\}$ and $B = \{3,4\}$

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), \}$$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2), \}$$

$$B \times A = \{(3, 1), (3, 2), (4, 1), (4, 2), \}$$

$$A \times \emptyset = \emptyset$$

Theorem 3.1. Let A, B, D and C be any sets in universal set X. Then

1. If $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$
2. $A \times B = B \times A \Leftrightarrow A = B$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
5. $A \times (B - C) = (A \times B) - (A \times C)$
6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
7. If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

Proof:1. Suppose that $A \times B \neq \emptyset$, this means $\exists (a, b) \in A \times B \Rightarrow a \in A \wedge b \in B$ (Def. of $A \times B$)
 $\Rightarrow A = \emptyset \wedge B = \emptyset$, this implies to C!
 $\therefore A \times B = \emptyset$

2. (\Rightarrow) Suppose that $A \times B = B \times A$ to prove $A = B$
 For each $(x, y) \in A \times B \Rightarrow x \in A \wedge y \in B$ (Def. of $A \times B$)
 $(x, y) \in B \times A \Rightarrow x \in B \wedge y \in A$ ($A \times B = B \times A$)
 $\Rightarrow A \subseteq B \wedge B \subseteq A$
 $\Rightarrow A = B$

(\Leftarrow) Suppose that $A = B$ to prove $A \times B = B \times A$
 Let $(x, y) \in A \times B \Leftrightarrow x \in A \wedge y \in B$ (Def. of $A \times B$)
 $\Leftrightarrow x \in B \wedge y \in A$ ($A = B$)
 $\Leftrightarrow (x, y) \in B \times A$
 $\therefore A \times B = B \times A$

3. To prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned}
 \text{Let } (x,y) \in A \times (B \cap C) &\Leftrightarrow x \in A \wedge y \in (B \cap C) && \text{(Def. of } A \times B) \\
 &\Leftrightarrow x \in A \wedge (y \in B \wedge y \in C) && \text{(Def. of } \cap) \\
 &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) && (\wedge \text{ distributives on } \wedge) \\
 &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in A \times C && \text{(Def. of Cartesian Product)} \\
 &\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C) && \text{(Def. of } \cap) \\
 \therefore A \times (B \cap C) &= (A \times B) \cap (A \times C)
 \end{aligned}$$

5. $A \times (B - C) = (A \times B) - (A \times C)$

$$\begin{aligned}
 \text{Let } (x,y) \in A \times (B - C) &\Leftrightarrow x \in A \wedge y \in (B - C) && \text{(Def. of Cartesian Product)} \\
 &\Leftrightarrow x \in A \wedge (y \in B \wedge y \notin C) && \text{(Def. of difference)} \\
 &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C) && \\
 &&& (\wedge \text{ distributives on } \wedge) \\
 &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \notin A \times C \\
 &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C) && \text{(dist. } \wedge \text{ on } \wedge) \\
 &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \notin A \times C && \text{(Def. of Cartesian Product)} \\
 &\Leftrightarrow (x,y) \in (A \times B) - (A \times C) && \text{(Def. of difference)} \\
 \therefore A \times (B - C) &= (A \times B) - (A \times C)
 \end{aligned}$$

6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

$$\begin{aligned}
 \text{Let } (x,y) \in (A \times B) \cap (C \times D) & \\
 &\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in C \times D && \text{(Def. of } \cap) \\
 &\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) && \text{(Def. of Cartesian Product)} \\
 &\Leftrightarrow (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) && (\wedge \text{ commut. and associ.}) \\
 &\Leftrightarrow x \in (A \cap C) \wedge y \in (B \cap D) && \text{(Def. of } \cap) \\
 &\Leftrightarrow (x,y) \in (A \cap C) \times (B \cap D) && \text{(Def. of Cartesian Product)}
 \end{aligned}$$

Definition 3.1. If A, A_1, A_2, \dots, A_n are sets in X , then

$$\prod_{i=1}^n A_i = A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n); x_1 \in A_1 \wedge x_2 \in A_2 \wedge \dots \wedge x_n \in A_n\}$$

$$A^n = A \times A \times \dots \times A$$

Examples: 3.2. 1) let $A = \{1,2\}, B = \{1\}$ and $C = \{3,4\}$. Then

$$A \times B \times C = \{(1,1,3), (1,1,4), (2,1,3), (2,1,4)\}$$

$$A^2 = A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$$

2) let R be a set of real numbers. Then

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Exercises

Exercise 1: Prove or disprove the following

1. $A \times B = A \times C \Leftrightarrow B = C$
2. $A \times (B \times C) = (A \times B) \times C$

Exercise 2: Prove or disprove the following

1. $P(A \times B) \subseteq P(A) \times P(B)$
2. $P(A \cup B) = P(A) \cup P(B)$

Exercise 3: Let $A = \{x \in \mathbb{N} : 2 \leq x < 4\}$, $B = \{2, 3, 6\}$ and $C = \mathbb{N}$.

Find $A \times (B \cup C)$, $A \times (B \cap C)$

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