Foundations of Mathematics

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Lecture 10.: Mathematical Foundations: Anti-symmetric relations , Equivalence Relations and Partitioning of sets

Introduction

Relations are a key concept in mathematics, used to describe connections between elements within a set. This lecture focuses on anti-symmetric relations, which limit mutual connections, and equivalence relations, which classify elements into distinct groups based on shared properties. These classifications naturally lead to partitioning sets, dividing a set into non-overlapping subsets. Together, these concepts provide a foundation for understanding mathematical structures and their applications.

Anti-Symmetric Relation:

A relation $\mathcal{R}$ on a set A is called anti-symmetric if for any $a, b \in A$ then $\mathcal{R}$ satisfies the statements

$$a{\sim}b \wedge b{\sim}a \Longrightarrow a = b$$

Examples: 3.15. let $A = \{1,2,3\}$. Are the following relations on a set A anti symmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$ not anti symmetric

$R_2 = \{(1,1)\}$ anti symmetric

$R_3 = \{(1,1), (1,2), (2,2), (3,3)\}$

$R_4 = \emptyset$

$R_5 = \{(1,1), (1,3), (3,2), (2,2), (3,3)\}$

Examples: 3.16. let $A = Z$. Are the following relations on a set A reflexive ? Symmetric? Anti-symmetric?

1) $a \sim b \Leftrightarrow a = b + 1$

Reflexive since for any a ∈ Z , $a \sim a \Leftrightarrow a \neq a + 1$ then the relation is not reflexive.

Symmetric since ∀a, b ∈ Z , $a \sim b \Leftrightarrow$ a = b + 1 but b $\neq a + 1$ then $(b, a) \notin R$ then the relation is not symmetric.

Anti-symmetric Since ∀a, b ∈ Z If $a \sim b \wedge b \sim a \Longrightarrow a = b + 1 \wedge b = a + 1 \Longrightarrow a = b$ then the relation is not anti-symmetric.

2) $a \sim b \Leftrightarrow a|b$

Reflexive since for any a ∈ Z , $a \sim a \Leftrightarrow a|a$ $(a = 1 \times a)$ then the relation is reflexive.

Symmetric since ∀a, b ∈ Z , $a \sim b \Leftrightarrow a|b$ but not nessary $b|a$

For example_ 2|4 but 4|2 then the relation is not symmetric.

Anti-symmetric Since ∀a, b ∈ Z If $a \sim b \wedge b \sim a \Longrightarrow a|b \wedge b|a \Longrightarrow \exists k_1, k_2 \in Z$ *such that* $b = k_1 a$ *and* $a = k_2 b \Longrightarrow k_1, k_2 = 1$

$\Longrightarrow$ *ether* $k_1 = k_2 = 1$ *or* $k_1 = k_2 = -1$

$\Longrightarrow a = b$ *or* $a = -b$ then the relation is not anti-symmetric.

3) $a \sim b \Leftrightarrow a - b = 2k\ , k \in Z$ (H.W.)

4) $a \sim b \Leftrightarrow a = 2b$

Theorem 3.6. : A relation $\mathcal{R}$ on a set A is anti-symmetric iff $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$.

Proof: $(\Longrightarrow)$ Suppose that $\mathcal{R}$ is anti-symmetric T.P. $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$

Let $(a, b) \in \mathcal{R} \cap \mathcal{R}^{-1} \Longrightarrow (a, b) \in \mathcal{R} \wedge (a, b) \in \mathcal{R}^{-1}$ (Def. of ∩)

$\Longrightarrow (a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R}$ (Def. of $\mathcal{R}^{-1}$)

$\Longrightarrow a = b$ ($\mathcal{R}$ is anti-symmetric)

$\Longrightarrow (a, b) \in I_A$

$\Longrightarrow (a, b) \in R \wedge (b, a) \in R$

$\Longrightarrow a = b$ ($\boldsymbol{\mathcal{R}}$ is anti symmetric)

$\Longrightarrow (a, b) \in I_A$

($\Leftarrow$) Suppose that $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$ T.P. $\boldsymbol{\mathcal{R}}$ is anti-symmetric

Let $(a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R} \Longrightarrow (a, b) \in \mathcal{R} \wedge (a, b) \in \mathcal{R}^{-1}$ (Def. of $\mathcal{R}^{-1}$)

$\Longrightarrow (a, b) \in \mathcal{R} \cap \mathcal{R}^{-1}$ (Def. of $\cap$)

$\Longrightarrow (a, b) \in I_A$ (By hyp.)

$\Longrightarrow a = b$

$\boldsymbol{\mathcal{R}}$ is anti-symmetric.

Equivalence Relation:

A relation $\mathcal{R}$ on a set A is called equivalence relation if and only if it satisfies reflexive, symmetric and transitive properties.

Examples: 3.16. let $A = Z$. Are the following relations on a set A Equivalence?

1) $a \sim b \Leftrightarrow a + b = 2k, k \in Z$

2) $a \sim b \Leftrightarrow a | b$

Solution: (H.W.)

Partition of a Set:

A collection of subsets $\{A_i : i \in J \subseteq N\}$ of A is called partition of A iff it satisfies the following conditions :

1. $A_i \neq \emptyset \ \forall i \in J$
2. $A_i \cap A_j = \emptyset \ \forall i \neq j$
3. $\bigcup_{i \in J} A_i = A$

Examples: 3.17. 1) let $A = Z$ and E and O are even and odd numbers respectively,

1. $E \neq \emptyset$ and $O \neq \emptyset$

2. $E \cap O = \emptyset$

3. $E \cup O = Z$

then $p = \{E, O\}$ is a partition of A

2) $A = (0,4)$ find Partition of A?

Let $A_1 = (0,1), A_2 = (1,2), A_3 = (2,3), A_3 = (3,4), A_5 = \{1,2,3\}$. Then

1. $A_i \neq \emptyset \quad \forall i \in J = \{1,2,3,4,5\}$

2. $A_i \cap A_j = \emptyset \quad \forall\, i \neq j$

3. $\bigcup_{i \in J} A_i = A$

3) find the another partition of A in 2) (H.W.)

4) $A = [-1,5)$ find three different Partitions of A?

Exercises

Exercise 1: Let $A = Z$ and define $a\, \mathcal{R}\, b \iff |a| = |b|, \forall a, b \in Z$

Is $\mathcal{R}$ symmetric? Anti-symmetric?

Exercise 2: Let $A = Z$ and define $a\, \mathcal{R}\, b \iff a = 1, \forall a \in Z$

$\mathcal{R} = \{(1, b); b \in Z\}$. Is $\mathcal{R}$ equivalence?

Exercise 3: Let $R = \{(a, b) \in Z \times Z : b = 1 - a\}$. Is R symmetric ? Anti-symmetric? .

Exercise 4: Let $A = R$ and define $a \sim b \iff ab \leq 0 \quad \forall a, b \in R$

Is the relation equivalence?

Exercise 5: Let $A = Z$ and $R = \{(a, b) \in Z \times Z : a - b = 5k, k \in Z\}$. Show that R is an equivalence relation?

References

1. Smith, P. (2003). *Introduction to Mathematical Logic*. Cambridge University Press. ISBN: 9780521008044.
2. Enderton, H. B. (1977). *Elements of set theory*. Academic press.
3. Shoenfield, J. R. (2000). *Mathematical Logic*. A K Peters. ISBN: 9781568811352.
4. Levy, A. (2012). *Basic set theory*. Courier Corporation..
5. Fraenkel, A. (1973). *Foundations of Set Theory*. Amsterdam: North-Holland.