Foundations of Mathematics

Instructor:
Uday Jabbar Quaez and Ruqaya Ibrahim
Academic Year: 2024-2025
Department of Mathematics, College of Education, Al-Mustansiriyah University

Lecture 13.: Inverse and Composition Mappings

Introduction

In mathematics, a function is a rule that assigns each element from a set (called the domain) to exactly one element in another set (called the codomain). The concept of an inverse function or inverse mapping is crucial in understanding how we can reverse this assignment process.

Inverse Mapping (Function):

Given a function $\mathrm{f}: \mathrm{A} \rightarrow \mathrm{B}$, an inverse mapping (or inverse function) of f, denoted as f^{-1}, is a function from B back to A such that:.

$$\mathrm{f}^{-1}(y) = \mathrm{x} \Leftrightarrow \mathrm{f}(x) = y$$

Examples: 4.5. Let $f: \{1,2,3\} \rightarrow \{a, b, c\}$;

$f = \{(1, a), (2, b), (3, c)\}$. *Then,* $f^{-1}: \{a, b, c\} \rightarrow \{1,2,3\}$

$f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

Remark (4.1): *Mathematically, the inverse of mapping exists iff the mapping is bijective.*

Examples: 4.6. 1) Let $f: Z \rightarrow Z$; $f(x) = x + 1$ find inverse of f?

Solution: firstly, to prove f is bijective

Let $\mathrm{x}_1, \mathrm{x}_2 \in Z$ such that $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2) \Longrightarrow \mathrm{x}_1 + 1 = \mathrm{x}_2 + 1 \Longrightarrow \mathrm{x}_1 = \mathrm{x}_2$

Then f is (1-1) ....(1)

Let $y \in Z \Longrightarrow \mathrm{y} - 1 = \mathrm{x} \in Z \Longrightarrow \mathrm{f}(x) = y$ then f is onto ....(2)

From (1)and (2) f is bijective

f^{-1} *exists* $\Longrightarrow f^{-1}\colon\ Z \ \to\ Z$ such that $f^{-1}(y) = y - 1$

2) Let $f\colon R \ \to\ R;\quad f(x) = (x-1)^2 - 1$. Is f^{-1} exists?

Solution:

Since $\mathrm{f}(-1) = \mathrm{f}(3) = 3$ but $-1 \neq 3$ this means that f is not (1-1). Hence f is not bijective and f^{-1} is not exists.

3) Let $f\colon R \ \to\ R;\quad f(x) = x^3$. Is f^{-1} exists ? if it exists find inverse of f?

Solution: firstly, to check f is bijective

Let $\mathrm{x}_1, \mathrm{x}_2 \ \in R$ such that $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2) \Longrightarrow \mathrm{x}_1^3 = \mathrm{x}_1^3 \Longrightarrow \mathrm{x}_1 = \mathrm{x}_2$

Then f is (1-1) ....(1)

Let $y \in R \Longrightarrow \sqrt[3]{\mathrm{y}} = \mathrm{x} \in R \Longrightarrow \mathrm{f}(x) = y$ then f is onto ....(2)

From (1)and (2) f is bijective

f^{-1} *exists* $\Longrightarrow f^{-1}\colon\ R \ \to R$ such that $f^{-1}(y) = \sqrt[3]{\mathrm{y}}$

4) Let $f\colon [0,\infty) \ \to [0,\infty);\quad f(x) = \sqrt{x}$. Is f^{-1} exists ? if it exists find inverse of f?

Solution: firstly, to check f is bijective

Let $\mathrm{x}_1, \mathrm{x}_2 \ \in [0,\infty)$ such that $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2) \Longrightarrow \sqrt{\mathrm{x}_1} = \sqrt{\mathrm{x}_2} \ \Longrightarrow \mathrm{x}_1 = \mathrm{x}_2$

Then f is (1-1) ....(1)

Let $y \in [0,\infty) \Longrightarrow y^2 = \mathrm{x} \in [0,\infty) \Longrightarrow \mathrm{f}(x) = y$ then f is onto ....(2)

From (1)and (2) f is bijective

f^{-1} *exists* $\Longrightarrow f^{-1}\colon\ [0,\infty) \ \to [0,\infty)$ such that $f^{-1}(y) = y^2$

Remark (4.2): *The following statements are hold:*

1. $(f^{-1})^{-1} = f$ (Check)
2. $f = f^{-1} \Leftrightarrow f = I_A$ (Check)

Definition (4.2): Let f and g be two mappings. Then, f *and* g are called equal denoted by $f = g$ if and only if $D_f = D_g$ *and* $f(x) = g(x)$, $\forall \mathbf{x} \in D_f$

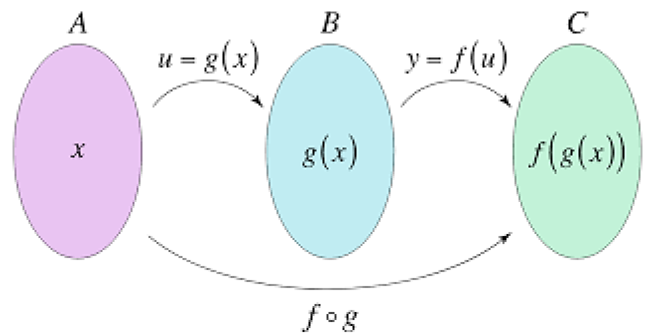
Composition of Mappings (Functions):

Let $g: A \to B$ and $f: B \to C$ be two mappings. Then, the composition mapping $fog: A \to C$ is defined as follows

$$\boldsymbol{fog(x) = f\big(g(x)\big)}, \qquad \forall \mathbf{x} \in \mathbf{A}$$

Remark (4.3): Let $f: A \to B$ and $g: C \to D$. Then

1. fog is exist if and only if $R_g \subseteq D_f$
2. gof is exist if and only if $R_f \subseteq D_g$
3. $fog \neq gof$



Examples: 4.7. 1) Let $f: Z \to Z$; $f(x) = x + 1$ and $g: Z \to Z$; $g(x) = 2x$
Is fog exist ? If it is exist, find fog?

Solution: Since $R_g \subseteq D_f \Longrightarrow fog$ is exist.

$$fog(x) = f\big(g(x)\big) = f(2x) = 2x + 1$$

2) Let $f: \left[0, \frac{\pi}{2}\right] \to [0,1]$; $f(x) = \sin(x)$ and $g: R \to R$; $g(x) = 2x$ Are fog and gof exist ?

Solution: Since $R_g = \mathrm{R}$ is not subset of $D_f = \left[0, \frac{\pi}{2}\right] \Longrightarrow fog$ is not exist.

Since $R_f = [0,1]$ is a subset of $D_f = R \Longrightarrow gof$ is exist.

$$gof(x) = g\big(f(x)\big) = g(\sin(x)) = 2\sin(x)$$

Theorem (4.4): Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two mappings, then

1. If f *and* g *are* $(1-1)$ then gof *is also* $(1-1)$
2. If gof is $(1-1)$ then f is $(1-1)$
3. . If f *and* g are onto then gof is also onto
4. If gof is onto then g is onto

Proof: 1. Suppose that f *and* g *are* $(1-1)$ to prove gof *is* $(1-1)$

Let $x_1, x_2 \in A$ such that $gof(x_1) = gof(x_2) \Longrightarrow g(f(x_1)) = g(f(x_2))$

$\Longrightarrow f(x_1) = f(x_2)$ (Since g is $(1-1)$)

$\Longrightarrow x_1 = x_2$ (Since f is $(1-1)$)

Then, gof is $(1-1)$

2. Suppose that gof is $(1-1)$ to prove f *is* $(1-1)$

Let $x_1, x_2 \in A$ such that

$f(x_1) = f(x_2) \Longrightarrow g(f(x_1)) = g(f(x_2))$ (Since g is mapping (well defined)

$\Longrightarrow gof(x_1) = gof(x_2)$ (Since gof is $(1-1)$)

$\Longrightarrow x_1 = x_2$

Then, f is $(1-1)$

3. Suppose that f *and* g are onto to prove gof is onto

Let $z \in C \Longrightarrow \exists y \in B$ *such that* $z = g(y)$ (Since g is *onto*)

Since $y \in B \Longrightarrow \exists x \in A$ *such that* $y = f(x)$ (Since$f$ is *onto*)

Then, $z = g(y) = g(f(x)) = gof(x)$ (Definition of composition)

Then gof is onto

4. Suppose that gof is onto then g is onto

Let $z \in C \implies \exists x \in A \text{ such that } z = gof(x)$ (Since gof is $onto$)

$\implies z = g(f(x)) \text{ and } y = f(x) \in B$ (Since f is $mapping$)

Then, $z = g(y)$

Then g is onto

Theorem (4.5): Let $f: A \to B$ and $I_A: A \to A$ (identity mapping) be mappings, then

1. $fo\, I_A = f$
2. $I_A of = f$ (H.W.)

Proof: 1. To prove $D_f = D_{I_A}$ $and\ f(x) = I_A(x), \forall \mathbf{x} \in D_f$

$D_f = D_{I_A}$ (Definition of identity mapping)

Let $x \in A \implies fo\, I_A(x) = f(I_A(x)) = f(x)$

Theorem (4.6): Let $A, B \text{ and } C$ are non empty sets. Then,

1. If $f: A \to B$ is bijective mapping then $f^{-1}of = I_A$ and $fof^{-1} = I_B$
2. If $f: A \to B$ is bijective mapping then f^{-1}: B $\to$ A is also bijective mapping.
3. If $f: A \to B, g: B \to C, h: C \to D$ then $(hog)of = ho(gof)$
4. If $f: A \to B$ and $g: B \to C$ are bijective mappings then $(gof)^{-1} = f^{-1}og^{-1}$

Proof: 1. To prove $D_{f^{-1}of} = D_{I_A}$ $and\ f^{-1}of\,(x) = I_A(x), \forall \mathbf{x} \in D_{I_A}$

$D_{f^{-1}of} = D_f = A$ (Definition of composition)

$D_{I_A} = A \implies D_{f^{-1}of} = D_{I_A}$

Let $x \in A \Longrightarrow f^{-1}of\,(x) = f^{-1}(f\,(x))$ ... (1)

Since f is a mapping then $\exists y \in B$ such that $y = f(x)$ ... (2)

Substitute (2) in (1) we have,

$f^{-1}of\,(x) = f^{-1}\big(f\,(x)\big) = f^{-1}(y) = x$ (Definition of inverse mapping)

$\therefore f^{-1}of = I_A$

In the similar method, prove $fof^{-1} = I_B$

2. Suppose that $f: A \to B$ is bijective mapping to prove f^{-1}: B → A is bijective mapping

- Since f is bijective mapping then f^{-1} is exist

Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$

Since f is onto $\exists x_1, x_2 \in A$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$

$\Longrightarrow x_1 = x_2$ (Since $f^{-1}(y_1) = f^{-1}(y_2)$)

$\Longrightarrow f(x_1) = f(x_2)$ (Since f is *well defined*)

$\Longrightarrow y_1 = y_2$

Then, f^{-1} is $(1-1)$. ...(1)

- Let $x \in A \Longrightarrow \exists y \in B$ *such that* $y = f(x)$ (Since f is *onto*)
$\Longrightarrow x = f^{-1}(y)$ (Definition of inverse mapping)
Then, f^{-1} is onto. ...(2)

From (1) and (2) f^{-1} is bijective mapping

3. H.W.
4. Suppose that $f: A \to B$ and $g: B \to C$ are bijective mappings to prove

$(gof)^{-1} = f^{-1}og^{-1}$

Let $h = gof: A \to C \Longrightarrow hoh^{-1} = I_C$ (from Theorem (4.6))
$ho(f^{-1}og^{-1}) = (gof)o(f^{-1}og^{-1})$
$= go(fof^{-1}og^{-1})$ (o is associative)
$= go(I_B og^{-1})$ (from Theorem (4.6))

$= gog^{-1}$ (from Theorem (4.5))

$= I_C$

Then, $hoh^{-1} = ho(f^{-1}og^{-1}) \Longrightarrow h^{-1} = (f^{-1}og^{-1})$

Exercises

Exercise 1: prove or disprove

1. If f *and* g *are* $(1-1)$ then $f^{-1}og^{-1}$ is $(1-1)$
2. If f *and* g *are bijective* then $f^{-1}og^{-1}$ is *bijective*
3. If f *is bijective* then $f^{-1}of$ is *bijective*

Exercise 2: Can a mapping have an inverse if it is not onto? Why?

Exercise 3: Find the formula for the inverse function $f^{-1}(x)$ of the following functions:

1. $\boldsymbol{f(x) = \frac{x}{2+x}}$
2. $\boldsymbol{f(x) = x^2 + 6x + 9}$
3. $\boldsymbol{f(x) = \sqrt[3]{x-1}}$

Exercise 4: If f *and* g are two bijective mappings such that $fo\, g = gof$

Does it follow that $f = g$? Why or why not?

Exercise 5: If f *and* g are two bijective mappings such that $h = gof$.

Show that f^{-1} **is well defined.**

References

1. Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal*, *20*(4), 282-300.
2. Enderton, H. B. (1977). *Elements of set theory*. Academic press.
3. Shoenfield, J. R. (2000). *Mathematical Logic*. A K Peters. ISBN: 9781568811352.
4. Halmos, P. R. (1960). Naive set theory. van Nostrand..
5. Fraenkel, A. (1973). *Foundations of Set Theory*. Amsterdam: North-Holland.
6. Rudin, W. (1964). *Principles of mathematical analysis* (Vol. 3). New York: McGraw-hill.