Foundations of Mathematics

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Lecture 14.: Direct Image Inverse Image

Introduction

In set theory and functions, the concepts of direct image and inverse image play a crucial role in understanding how functions map elements between sets. These concepts are fundamental in mathematical analysis, topology, and algebra.

Direct Image:

Given a function $\mathrm{f}: \mathrm{A} \rightarrow \mathrm{B}$, The direct image of $C \subseteq A$ under f denoted as $f(C)$ is defined as

$$f(C) = \{y \in B ; y = f(x), x \in C\}$$

Examples: 4.8. Let $f: \{1,2,3\} \rightarrow \{a, b, c\}$;

$f = \{(1, a), (2, b), (3, c)\}$ *and* $C = \{1,2\}$*. Then,* $f(C) = \{a, b\}$

Theorem (4.7): Let $f: A \rightarrow B$ be a mapping, and $C, D \subseteq A$. Then,

1. If $C \subseteq D$ then $f(C) \subseteq f(D)$
2. $f(C \cap D) \subseteq f(C) \cap f(D)$
3. If f is an $(1-1)$ then $f(C \cap D) = f(C) \cap f(D)$
4. $f(C \cup D) \subseteq f(C) \cup f(D)$
5. $f(C) - f(D) \subseteq f(C - D)$
6. $f(C - D) \subseteq f(C)$

Proof: 1. Suppose that $C \subseteq D$ to prove $f(C) \subseteq f(D)$

Let $y \in f(C)$, then $\exists x \in C$ *such that* $y = f(x)$ $(Def. of\ direct\ image)$

But $C \subseteq D \Longrightarrow x \in D, y = f(x) \Longrightarrow y \in f(D)$

2. To prove $f(C \cap D) \subseteq f(C) \cap f(D)$

Let y $\in f(C \cap D)$, then $\exists x \in C \cap D$ *and* $y = f(x)$ $(Def. of\ direct\ image)$

$\Longrightarrow x \in C \wedge x \in D$ *such that* $y = f(x)$

$\Longrightarrow x \in C\ ;\ y = f(x) \wedge x \in D;\ y = f(x)$

$\Longrightarrow$ y $\in f(C) \wedge$ y $\in f(D)$

3. Suppose that f is an $(1-1)$ to prove $f(C \cap D) = f(C) \cap f(D)$

From 2. we have $f(C \cap D) \subseteq f(C) \cap f(D)$ .... (1)

We need only to prove $f(C) \cap f(D) \subseteq f(C \cap D)$

Let y $\in f(C) \cap f(D) \Longrightarrow$ y $\in f(C) \wedge$ y $\in f(D)$

$\Longrightarrow \exists x_1 \in C; y = f(x_1) \ \wedge \ \exists x_2 \in D; y = f(x_2)$$(Def. of\ direct\ image)$

$\Longrightarrow$ Since $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2 = x$ *(By hyp. f is an* $(1-1)$*)*

$\Longrightarrow x \in C \cap D$ *such that* $y = f(x) \Longrightarrow$ y $\in f(C \cap D)$

$f(C) \cap f(D) \subseteq f(C \cap D)$ .... (2)

From (1) and (2), $f(C \cap D) = f(C) \cap f(D)$

then $\exists x \in C \cap D$ *and* $y = f(x)$ $(Def. of\ direct\ image)$

$\Longrightarrow x \in C \wedge x \in D$ *such that* $y = f(x)$

$\Longrightarrow x \in C\ ;\ y = f(x) \wedge x \in D;\ y = f(x)$

$\Longrightarrow$ y $\in f(C) \wedge$ y $\in f(D)$

4. , 5. And 6. (H.W.)

Inverse Image:

Given a function $\mathrm{f}: \mathrm{A} \rightarrow \mathrm{B}$, The inverse image of $E \subseteq B$ under f denoted as $f^{-1}(E)$ is defined as

$$f^{-1}(E) = \{x \in A;\ f(x) \in E\}$$

Examples: 4.9. **1)** Let $f: \{1,2,3\} \rightarrow \{a, b, c\}$;

$f = \{(1, a), (2, b), (3, c)\}$ *and* $E = \{a, b\}$. *Then,* $f^{-1}(E) = \{1,2\}$

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2) Let $f: N \rightarrow N$; $f(x) = x^2 + 1$ and $E = \{1,2,3\}$. *Then,* $f^{-1}(E) = \{1\}$

Since

$$f^{-1}(E) = \{x \in N;\ f(x) = x^2 + 1 \in E\}$$

$1 \in E \Longrightarrow x^2 + 1 = 1 \Longrightarrow x = 0 \notin N$

$2 \in E \Longrightarrow x^2 + 1 = 2 \Longrightarrow x = 1 \in N$

$3 \in E \Longrightarrow x^2 + 1 = 3 \Longrightarrow x = \sqrt{2} \notin N$

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3) Let $f: R \rightarrow R$; $f(x) = x^2 - 2$ and $E = \{2,7\}$. *Then,* $f^{-1}(E) = \{2, -2, 3, -3\}$

Since

$2 \in E \Longrightarrow x^2 - 2 = 2 \Longrightarrow x = \mp 2 \in R$

$7 \in E \Longrightarrow x^2 - 2 = 7 \Longrightarrow x = \mp 3 \in R$

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4) Let $f: R \rightarrow R$; $f(x) = \sqrt{x^2 + 5}$ and $E = \{-1,0,1\}$. *Then,* $f^{-1}(E) = \emptyset$

Since $-1 \in E \Longrightarrow \sqrt{x^2 + 5} = -1 \Longrightarrow x = \sqrt{-4} \notin R$

$0 \in E \Longrightarrow \sqrt{x^2 + 5} = 0 \Longrightarrow x = \sqrt{-5} \notin R$

$1 \in E \Longrightarrow \sqrt{x^2 + 5} = 1 \Longrightarrow x = \sqrt{-4} \notin R$
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Theorem (4.8): Let $f: A \to B$ be a mapping, $E_1, E_2 \subseteq B$ *and* $C \subseteq A$. Then,

1. $f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$
2. $f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2)$
3. If $E_1 \subseteq E_2$ then $f^{-1}(E_1) \subseteq f^{-1}(E_2)$
4. $f^{-1}(E_1 - E_2) = f^{-1}(E_1) - f^{-1}(E_2)$
5. $f^{-1}(E_1 - E_2) \subseteq f^{-1}(E_1)$
6. $C \subseteq f^{-1}(f(C))$
7. f is an $(1-1)$ if and only if $C = f^{-1}(f(C))$
8. $f(f^{-1}(C)) \subseteq C$
9. f is an *onto* if and only if $f(f^{-1}(C)) = C$

Proof: 1. To prove $f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$

Let $x \in f^{-1}(E_1 \cap E_2) \Leftrightarrow f(x) \in E_1 \cap E_2$ (Definition of f^{-1})

$\Leftrightarrow f(x) \in E_1 \wedge f(x) \in E_2$ (Definition of $\cap$)

$\Leftrightarrow x \in f^{-1}(E_1) \wedge x \in f^{-1}(E_2)$ (Definition of f^{-1})

$\Leftrightarrow x \in f^{-1}(E_1) \cap f^{-1}(E_2)$ (Definition of $\cap$)

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2. To prove $f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2)$

Let $x \in f^{-1}(E_1 \cup E_2) \Leftrightarrow f(x) \in E_1 \cup E_2$ (Definition of f^{-1})

$\Leftrightarrow f(x) \in E_1 \vee f(x) \in E_2$ (Definition of $\cap$)

$\Leftrightarrow x \in f^{-1}(E_1) \vee x \in f^{-1}(E_2)$ (Definition of f^{-1})

$\Leftrightarrow x \in f^{-1}(E_1) \cup f^{-1}(E_2)$ (Definition of $\cap$)

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3. Suppose that $E_1 \subseteq E_2$ to prove $f^{-1}(E_1) \subseteq f^{-1}(E_2)$

Let $x \in f^{-1}(E_1) \Longrightarrow f(x) \in E_1$ (Definition of f^{-1})

$\Longrightarrow f(x) \in E_2$ (By hyp. $E_1 \subseteq E_2$)

$\Longrightarrow x \in f^{-1}(E_2)$ (Definition of f^{-1})

Then $f^{-1}(E_1) \subseteq f^{-1}(E_2)$

4. To prove $f^{-1}(E_1 - E_2) = f^{-1}(E_1) - f^{-1}(E_2)$

Let $x \in f^{-1}(E_1 - E_2) \Leftrightarrow f(x) \in E_1 - E_2$ (Definition of f^{-1})

$\Leftrightarrow f(x) \in E_1 \wedge f(x) \notin E_2$ (Definition of $\cap$)

$\Leftrightarrow x \in f^{-1}(E_1) \wedge x \notin f^{-1}(E_2)$ (Definition of f^{-1})

$\Leftrightarrow x \in f^{-1}(E_1) - f^{-1}(E_2)$ (Definition of $\cap$)

5. To prove $f^{-1}(E_1 - E_2) \subseteq f^{-1}(E_1)$

Since $f^{-1}(E_1) - f^{-1}(E_2) \subseteq f^{-1}(E_1)$ ($A - B \subseteq A$)

By using 4. $f^{-1}(E_1 - E_2) = f^{-1}(E_1) - f^{-1}(E_2) \subseteq f^{-1}(E_1)$

Then, $f^{-1}(E_1 - E_2) \subseteq f^{-1}(E_1)$

6. To prove $C \subseteq f^{-1}\big(f(C)\big)$

Let $x \in C \Longrightarrow f(x) \in f(C)$ (f is mapping and def. of direct image)

$\Longrightarrow x \in f^{-1}\big(f(C)\big)$ (Definition of f^{-1})

Then, $C \subseteq f^{-1}\big(f(C)\big)$

7. $(\Longrightarrow)$ Suppose that f is an $(1-1)$ to prove $C = f^{-1}\big(f(C)\big)$

From 6. $C \subseteq f^{-1}(f(C))$, we only need to prove $f^{-1}(f(C)) \subseteq C$

Let $x \in f^{-1}(f(C)) \Rightarrow f(x) \in f(C)$ (Definition of f^{-1})

Assume that $\exists x^* \in A$ such that $f(x^*) \in f(C)$ and $f(x) = f(x^*)$

$\Rightarrow x = x^*$ *(f is (1-1))*

$\Rightarrow x \in f^{-1}(f(C))$

Then, $f^{-1}(f(C)) \subseteq C$

Thus, $f^{-1}(f(C)) = C$

($\Rightarrow$) Suppose that $C = f^{-1}(f(C))$ to prove f is an $(1-1)$

Assume that f *is not* $(1-1)$

$\exists x_1, x_2 \in A ;\ f(x_1) = f(x_2)$ *and* $x_1 \neq x_2$

Let $C = \{x_1\} \Rightarrow x_1 \in C \Rightarrow f(x_1) \in f(C)$ (def. of direct image)

$\Rightarrow f(x_2) \in f(C)$ $(f(x_1) = f(x_2))$

$\Rightarrow f(x_1) \in f(C)$ *and* $f(x_2) \in f(C)$

$\Rightarrow x_1 \in f^{-1}(f(C)) = C$ *and* $x_2 \in f^{-1}(f(C)) = C$

This leads to a contradiction , since $x_1 \neq x_2$

8. and 9. (H.W.)

Exercises

Exercise 1: prove or disprove Let $f: A \rightarrow B$ be a mapping, $C \subseteq A$ and $E \subseteq B$. Then,

1. $f^{-1}(f(C)) = C$
2. $f(f^{-1}(E)) = E$
3. If f *is a bijective* then $f^{-1}(f(C)) = C$ and $f(f^{-1}(E)) = E$

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