Foundations of Mathematics

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Lecture 16.: Equivalent of Sets

Introduction

In mathematics, the concept of sets is a fundamental building block for many mathematical branches. Among the associated concepts are set equivalence and the distinction between finite and infinite sets. These concepts help in understanding complex mathematical structures and dealing with infinite sets.

Cardinality of Finite Sets :

Let $J_m = \{1,2,\dots,m\}$ be a set. A set A is called finite of size m ($n(A) = m$) if and only if $A \approx J_m$. The positive number m is called *the cardinality of A.*

A is called finite $\mathrm{n(A)} = \mathrm{m} \Leftrightarrow A \approx J_m$

Examples: 5.5.

1) Let $A = \{a, b, c\}$. Then $A \approx J_m$ since

   If we choose $f = \{(a,1),(b,2),(c,3)\}$, f is bijective ($(1-1)$ *and onto*)

2)) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}\right\}$. Then is finite $\mathrm{n(A)} = 100$ *and* $A \approx J_{100}$ since

   Define a function $f: A \to J_m; f(x) = \frac{1}{x}$, then f is bijective (check)

Theorem (5.2): Any two finite sets have the same *cardinality* if and only if there exists a bijective mapping between them

Proof: Let A and B be two finite sets.

$\Longrightarrow$) Suppose that A and B have the same *cardinality* $n(A) = n(B) = m$ to prove $A \approx B$

Define a set $J_m = \{1,2,\dots,m\}$ then

$A \approx J_m$ *and* $B \approx J_m$ (A and B are finite have m *cardinality*)

$A \approx J_m$ *and* $J_m \approx B$ ($\approx$ *Symmetric*)

$A \approx B$ ($\approx$ *Transtive*)

$\Leftarrow$) Suppose that $A \approx B$ to prove A and B have the same *cardinality* $n(A) = n(B) = m$

Take $n(A) = m$ (A finite), then

$A \approx J_m$ and $A \approx B$ (def. *cardinality and assumption*)

$B \approx A$ and $A \approx J_m$ ($\approx$ *Symmetric*)

$B \approx J_m$ ($\approx$ *Transtive*)

$n(B) = m$

Theorem (5.3): Let A be a finite set and $A_1, A_2, \dots, A_n$, be a partition of A. Then

$$n(A) = n(A_1) + n\,(A_2) + \cdots n\;(A_n)$$

Theorem (5.4): Let A and B be finite sets. Then

$n(A \cup B) = n(A) + n\,(B) - n(A \cap B)$

Proof: The family $\{A - B, B - A, A \cap B\}$ represents a partition of $A \cup B$ (check)

By Theorem (5.3), we have

$n(A \cup B) = n(A - B) + n\,(B - A) + n(A \cap B)$ ***...... (1)***

The family $\{A - B, A \cap B\}$ represents a partition of A (check)

$n(A) = n(A - B) + n(A \cap B)$

$\Longrightarrow n(A - B) = n(A) - n(A \cap B)$ ***...... (2)***

The family $\{B - A, A \cap B\}$ represents a partition of B (check)

$n(B) = n(B - A) + n(A \cap B)$

$\Longrightarrow \; n(B - A) = n(B) - n(A \cap B)$ ***...... (3)***

Substituting equations (2) & (3) into equation (1), we obtain

$n(A \cup B) = n(A) - n(A \cap B) \quad + n(B) - n(A \cap B) + n(A \cap B)$

Theorem (5.5): Let A and B be two finite sets. Then

$$n(A \times B) = n(A).n(B)$$

Cardinality of Infinite Sets:

Let A be an infinite set. Then the cardinality of A is not a finite positive number. The cardinality of A is denoted by $\aleph_0$

Examples: 5.6.

1) Let $A = N$ Then $n(A) = \aleph_0$

2)) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \right\}$. Then A is infinite set and n(A) = $\aleph_0$

Definition (5.2): An infinite set A is called countable if it is equivalent to natural numbers set.

$$A \approx N \Leftrightarrow A \text{ is countable}$$

Examples: 5.7.

1) N is countable since $N \approx N$

2) Z^+ is countable since $Z \approx N$ ***(check)***

3) E is countable since $E \approx N$ ***(check)***

4) O is countable since $Z \approx N$ ***(check)***

5) Z is countable since $Z \approx N$ ***(check)***

Remark 5.6:

1) Every finite set is countable

2) Q the set of rational numbers is countably infinite set

3) R the set of real numbers is uncountable infinite set

Theorem (5.7): Any infinite subset of an infinite countable set is countable

Theorem (5.8): If A is countably infinite set then A$\cup \{a\}$ is also countably infinite set.

Proof: Since A is countably infinite set then $A \approx N$

$\exists f: A \rightarrow N \text{ is bijective}$

Define $g(x) = \begin{cases} 1 & , \quad x = a \\ f(x) + 1 & , \quad x \neq a \end{cases}$

Firstly, to prove g is a mapping

For each $x \in$ A $\cup \{a\}$, either $x = a \Longrightarrow g(x) = 1$

Or $x \neq a \Longrightarrow \exists y \in N, y = f(x) \Longrightarrow \exists y + 1 \in N, y + 1 = f(x) + 1 = g(x)$

g is closure

Let $x_1, x_2 \in A$; either $x_1 = x_2 \neq a \Longrightarrow f(x_1) = f(x_2)$ $(f \text{ is a mapping })$

$\Longrightarrow f(x_1) + 1 = f(x_2) + 1 \Longrightarrow g(x_1) = g(x_2)$

Or $x_1 = x_2 = a \Longrightarrow g(x_1) = g(x_2)$

g is well defined

Then, g is a mapping

Now, to prove g is a bijective (H.W.)

Exercise: Prove that the following sets are countable

1. $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \right\}$
2. $A = \left\{1, \sqrt{2}, \sqrt{3}, \dots, \right\}$
3. $A = \{e, e^2, \dots, \}$

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