Foundations of Mathematics

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Lecture 1: Introduction to the Foundations of Mathematics

Introduction

Mathematics is the study of numbers, patterns, and structures. It has practical applications in natural sciences, engineering, and various other fields. Mathematics is fundamental in the development of other scientific fields like physics, engineering, and computing.

Chapter One

- **Propositions (Statements)**
- **Mathematical Proof**
- **Quantifiers**

Definition 1.1. (Mathematical Logic**)**

Mathematical logic is the study of reasoning and the principles of valid inference in mathematics. A logical statement is a sentence that can be classified as either true or false.

Definition 1.2. (Proposition**)**

A proposition is a declarative statement that is either 'True' (T) or 'False' (F) but cannot be both true and false simultaneously. Letters such as p, q, r, and so forth are used to represent propositions.

Examples: 1.1.

- "The number 2 is a prime number" (True)
- "The number 5 is an even number" (False)
- " $x + y = 0$ " (Not a proposition)

Definition 1.3. (Negation of a Proposition**)**

Let p be a proposition. The negation of p is called 'not P' and is denoted by (~p).

p	~p
T	F
F	T

Examples: 1.2.

- " P: $3 < 5$" , "~p: $3 \geq 5$"
- "q: $2 + 1 = 5$" , " ~~q: $2 + 1 \neq 5$"
- "r: The square has four sides" , "~r: The square has not four sides"

Remark 1.1.

If p is a proposition, then $\sim(\sim p) = p$.

Compound Propositions:

Propositions are divided into two types:

1. **Primitive proposition:**

   If it cannot be divided into simpler propositions.

2. **Composite proposition:**

   If it is composed of more than one primitive proposition using logic connective operators.

Logic Connective Operators:

i. Conjunction Operator (and , ∧)

Let p and q are two propositions. the conjunction of p and q is denoted by "$p \wedge q$" and read "p and q". the table truth of the conjunction of p and q is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples: 1.3.

Find the truth value of the following statements:

a) "$2 + 2 = 4$ and $2 + 3 = 5$"
   $T \wedge T = T$

b) "3 is a prime number and π is a rational number "
   $T \wedge F = F$

c) "p: $x + y = y + x, x, y \in R$ and q: $2 > 10$"
   $T \wedge F = F$

Properties of conjunction operator (and , ∧)

Let p, q, r be any propositions. Then

1. $p \wedge q = q \wedge p$ (commutative)
2. $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ (associative)
3. $p \wedge p = p$ (idempotent law)
4. $p \wedge T = p$ (identity law)
5. $p \wedge F = F$
6. $p \wedge \sim p = F$

The proof of (1-6) by using the truth table of "and".

ii. Disjunction Operator (Or ,∨)

Let p and q are two propositions. the disjunction of p and q is denoted by "$p \vee q$" and read "p or q". the table truth of the conjunction of p and q is

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Examples: 1.4.

Find the truth value of the following statements:

a) "2 + 2 = 4 or 2 + 3 = 5"
   T ∨T=T
b) "3 is a prime number and π is a rational number "
   T ∨F=T
c) "p: $x + y = y + x, x, y \in R$ and q: 2 > 10"
   T ∨F=T

Properties of disjunction operator (or ,∨)

Let p, q, r be any propositions. Then

1. $p \vee q = q \vee p$ (commutative)
2. $(p \vee q) \vee r = p \vee (q \vee r)$ (associative)
3. $p \vee p = p$ (idempotent law)
4. $p \vee T = T$ (identity law)
5. $p \vee F = p$
6. $p \vee \sim p = T$

The proof of (1-6) by using the truth table of "or".

Truth Table for the Conditional Operator:

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

- If the antecedent is true and the consequent is true, the conditional statement is true.
- If the antecedent is true but the consequent is false, the conditional statement is false.
- If the antecedent is false, the conditional statement is true regardless of the consequent.

Examples: 1.5.

If p: "The number 4 is even" and q: "The number 2 is prime," then the conditional statement p → q says "If the number 4 is even, then the number 2 is prime." In this case, the statement is true because both the antecedent and the consequent are true.

Remark 1.2.

1) In the conditional statement "p → q", P is called the hypothesis and q is called the conclusion. The interpretation of p → q is as follows:
   a. If p is true, then q is true.
   b. q is true if p is true.
   c. q is true only if p is true.
   d. p is a sufficient condition for q.
   e. q is a necessary condition for p.
2) Converse, Inverse, and Contrapositive

The converse of p → q is q → p. The inverse of p → q is ~P → ~q, and the contrapositive is ~q → ~p.

Example: 1.5.

Find the truth value of the following statements:
1. If 2 is a prime number, then 2 is an odd number.
Truth value: False

2. If -1 = 1, then $(-1)^2 = (1)^2$.
Truth value: True

Properties of Conditional Operator(if then , →)

Let p, q, and r be propositions. Then, the following properties hold:

1. (p → q) ≠ (q → p)
2. (p → q) → r ≠ p → (q → r)

Bi-Conditional Operator

The bi-conditional operator, represented as p ↔ q, is defined as the logical equivalence of p and q. That is, p ↔ q is true if and only if both p and q are either true or false together.

Truth Table for the Bi-Conditional Operator:

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

Example: 1.6.(a)

Find the truth value of the following bi-conditional statement:

1. $(x > 0) \leftrightarrow (2x > 0)$

If $x > 0$, then $2x > 0$: True

If $x \leq 0$, then $2x \leq 0$: True
Therefore, the truth value is: True

Properties of Bi-Conditional Operator(if and only if , ↔)

Let p, q, and r be propositions. Then, the following properties hold:

1. $(p \leftrightarrow q) = (q \leftrightarrow p)$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$

Example: 1.6.(b)

Wright the truth table of these statements $(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \to \sim \boldsymbol{q})$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$\boldsymbol{p} \to \boldsymbol{q}$	$\sim \boldsymbol{p} \to \sim \boldsymbol{q}$	$(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \to \sim \boldsymbol{q})$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Exercises

Exercise 1: Wright the truth table of these statements

a) $\sim p \vee q$
b) $\sim \boldsymbol{p} \vee (\boldsymbol{p} \wedge \sim \boldsymbol{q})$

Exercise 2: Is the statement "The number 7 is a prime number" true or false? Explain your answer.

Exercise 3: Let p, q, r be propositions such that

P: 29 is a prime number , q: $x - x = 0$, r: $-3 \in N$. Find the truth value of the following statements:

a) $(p \vee q) \vee r$
b) $\sim q \vee r$
c) $\sim(\sim p \vee q)$
d) $(p \wedge q) \vee (p \vee r)$

Exercise 4: Evaluate the truth value of the following logical expressions:

1) $((5 > 3)\ and\ (8 = 8))\ or\ ((7 < 2)\ and\ (4 > 1))$
2) $((10 = 10)\ and\ (12 <= 15))\ or\ ((5 >= 6)\ and\ (7 < 7))$
3) $(false\ or\ (true\ and\ false))\ and\ (true\ or\ (false\ and\ true))$
4) $((3 < 9)\ and\ (6 >= 6))\ or\ (not\ (5 = 5))$
5) $(true\ and\ (false\ or\ (true\ and\ false)))\ and\ (true\ or\ false)$

Exercise 5:. Determine if the second statement is a logical consequence of the first statement:

1) "If it rains, the ground will be wet." and "The ground is wet." — Is it a logical consequence that "It rained"?
2) "If I study, I will pass the exam." or "If I do not study, I will fail." — Is it a logical consequence that "I will pass if I study"?
3) "All cats are animals." and "Fluffy is a cat." — Is it a logical consequence that "Fluffy is an animal"?

Exercise 6:. Construct the truth table for the following expressions and determine whether they are tautologies, contradictions, or contingencies:

a. (p and (q or r)) or (not q and not r)
b. ((p or q) and (not p or r)) and (p or (q and not r))
c. (p or q) and (not (p and q))
d. (p and not q) or (not p and q)

Exercise 7: Find the truth value of the following

[(If $2+3=4$ then $x+4=4+x$) and 8 is an even number] if and only if [$2 \leq -10$ or $2 \geq -10$]

Exercise 8: Let p be a propositions. Then find the truth value of the following

p ↔T , p ↔F, p ↔~p , p ↔ p

Exercise 9: Let P: 29 is a prime number , q: $x - x = 0$, r: $-3 \in N$. Find the truth value of the following statements:

a) $(p \leftrightarrow q)$
b) $(p \leftrightarrow r) \vee q$
c) $(p \leftrightarrow q) \vee (q \leftrightarrow \sim r)$

Exercise 10: Evaluate the truth value of the following logical expressions:

1) $if\ ((-4 > 3) then\ (3 = 3))\ or\ (if\ (5 < 2)\ then\ (4 > 1))$
2) $\big((10 = 10) and\ (12 <= 15)\big) if and\ only\ if\ \ ((5 >= 6)\ and\ (7 < 7))$
3) $(false\ \ iff\ \ (true\ and\ false))\ and\ (if\ true\ then\ \ (false\ and\ true))$

Exercise 11:. Construct the truth table for the following statements:

a. ~p ∧ q
b. (p ∧ q) → (p ∨ q)
c. (p ⟶ q) ∨ ~ (q ↔ p)
d. $\boldsymbol{p} \vee (\sim \boldsymbol{q}) \leftrightarrow \sim \boldsymbol{p} \wedge \boldsymbol{q}$
e. $[(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \longrightarrow \boldsymbol{r})] \longrightarrow (\boldsymbol{p} \wedge \sim \boldsymbol{r})$
f. $[\boldsymbol{p} \longrightarrow \boldsymbol{q}] \leftrightarrow (\sim \boldsymbol{q} \longrightarrow \boldsymbol{p} \wedge \boldsymbol{q})$

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