Foundations of Mathematics

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Lecture 2.: Logical Equivalence

Introduction

In the study of logic, propositions are statements that can either be true or false. Logical equivalence is an essential concept in propositional logic, which states that two propositions have the same truth value across all possible truth assignments. Understanding logical equivalence is fundamental to simplifying logical expressions and proving their validity.

In this lecture, we will explore the definitions and properties of logical equivalence, including common laws such as De Morgans Laws and the Distributive Laws. We will also practice simplifying logical statements using truth tables and laws of logical equivalence. Additionally, we will explore different methods of proving mathematical statements, including direct proof and proof by contradiction. We will also examine conditional statements and introduce the concept of open sentences and truth sets, all of which are crucial for understanding how mathematical assertions are validated.

Definition of Logical Equivalence:

Two statements (propositions) that have the same truth values are called logically equivalent. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

Examples: 1.7.

Show that $\sim(p \lor q) = \sim p \land \sim q$ (logically equivalent).

Hint: Use a truth table.

Solution:

p	q	p ∨ q	~(p ∨ q)	~p	~q	~p ∧ ~q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Definition 1.4. Let p, q, and r be propositions. Then, define the following

1. $(p \wedge q) = \sim(\sim p \vee \sim q)$
2. $(p \to q) = \sim p \vee q$
3. $(p \leftrightarrow q) = (p \to q) \wedge (q \to p)$

De Morgans Laws

Theorem 1.3. (*De Morgans Laws*)

These laws allow the negation of conjunctions and disjunctions to be transformed:

1. $\sim(p \wedge q) = \sim p \vee \sim q$ *(H.W)*
2. $\sim(p \vee q) = \sim p \wedge \sim q$ *(H.W)*

Examples: 1.8. Simplify the following statements using laws of logical equivalence:

1. $\sim(p \wedge \sim q)$
2. $\sim(\sim p \to q)$
3. $\sim(\sim p \leftrightarrow q)$ *(H.W)*

Solution: Use De Morgans law and implications definition for step-by-step simplification

1.

$$\sim(p \wedge \sim q) = \sim p \vee \sim\sim q \qquad (De\ Morgans\ Laws)$$

$$\sim(p \wedge \sim q) = \sim p \vee q \qquad (since\ \ \sim\sim q = q)$$

$$\sim(p \wedge \sim q) = p \to q \qquad (def. \to)$$

2.

$$\sim(\sim p \to q) = \sim(\sim\sim p \vee q) \qquad (def. \to)$$

$$= \sim(p \vee q) \qquad (\sim\sim p = p)$$

$$= \sim p \wedge \sim q \qquad (De\ Morgans\ Laws)$$

Important Laws of Logical Equivalence:

1. ***Commutative Law:***

$$p \wedge q = q \wedge p$$
$$p \vee q = q \vee p$$
$$p \leftrightarrow q = q \leftrightarrow p$$

2. *Associative Law:*

$$(p \wedge q) \wedge r = p \wedge (q \wedge r)$$

$$(p \vee q) \vee r = p \vee (q \vee r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

3. *Distributive Law: (left)*

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

$$p \vee (q \rightarrow r) = (p \vee q) \rightarrow (p \vee r)$$

$$p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$$

(Right) in the same equivalents on the side right**.**

4. *Idempotent Law:*

$$p \wedge p = p$$
$$p \vee p = p$$

5. *Identity Law:*

$$p \wedge T = p$$
$$p \vee F = p$$

Examples: 1.9. Simplify the following statements using laws of logical equivalence:

1. $\sim p \vee (p \wedge \sim q)$
2. $\sim(\sim p \rightarrow q) \vee q$ *(H.W)*
3. $p \vee (\sim p \leftrightarrow q)$ *(H.W)*

Solution:

1.

$$\sim p \vee (p \wedge \sim q) = (\sim p \vee p) \wedge (\sim p \vee \sim q) \qquad (Distributive\ Law)$$
$$= (T) \wedge (\sim p \vee \sim q) \qquad (since\ \sim p \vee p = T)$$

$$= (\sim p \vee \sim q) \qquad (Identity\ Law)$$

Examples: 1.10. Prove that the following statement:

$$\sim(p \vee (\sim p \wedge q) = \sim p \wedge \sim q$$

Solution:

$$\sim(p \vee (\sim p \wedge q) = \sim p \wedge \sim(\sim p \wedge q) \qquad (De\ Morgans\ Laws)$$

$$= \sim p \wedge (\sim\sim p \vee \sim q) \qquad (De\ Morgans\ Laws)$$

$$= \sim p \wedge (p \vee \sim q) \qquad (\ since\ \sim\sim p = p\ )$$

$$= (\sim p \wedge p\ ) \vee (\sim p \wedge \sim q) \qquad (Distributive\ Law)$$

$$= (F\ ) \vee (\sim p \wedge \sim q) \qquad (since\ \sim p \vee p = T)$$

$$= (\sim p \wedge \sim q) \qquad (Identity\ Law)$$

Methods of Proving Mathematical Statements (or Theorems):

1. Direct Proof of a conditional statement p → q

A direct proof is a logical process that starts from the assumption of the hypothesis and works step by step to reach the conclusion.

Definition 1.5. An integer x is called even if there exists k such that $x = 2k$.

Definition 1.6. An integer x is called odd if there exists k such that $x = 2k + 1$.

Theorem 1.4. If x is an odd natural number, then x^2 is odd.

Proof: Assume that x is an odd natural number. We must prove x^2 is odd.

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$$x^2 = x.x = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

Theorem 1.5. (Homework): If x is an even natural number, then x^2 is even.

Theorem 1.6. (Homework): The sum of two even natural numbers is even.

2. Direct Proof of a conditional statement p ↔ q

To prove a proposition in the form p ↔ q, we prove its equivalence. i.e.,

$$\boldsymbol{p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)}$$

Theorem 1.7. x is odd number if and only if $x + 1$ is an even number.

Proof: Let p: x is odd number and q: x+1 is an even number

To prove p → q and q → p

Prove p → q

Let $x \in O$ then $x = 2k + 1, k \in Z$

$x + 1 = 2k + 2 = 2(2k + 1)$, $k + 1 \in Z$

Assume that $r = k + 1$ then $x + 1 = 2r$;, $r \in Z$

$$x + 1 \in E$$

Prove q → p:

Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k, k \in Z$

$x = 2k - 1$ ... (1)

Since $k \in Z$, then $r = k - 1 \in Z$

$$k = r + 1 \quad \text{... (2)}$$

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1 \in O$

Theorem 1.8. (Homework): x is an even number if and only if x^2 is even number.

Theorem 1.9.(a) (Homework): x is an odd number if and only if x^2 is odd number.

Proof by Contradiction

Proof by contradiction involves assuming the opposite of what you want to prove, and then showing that this assumption leads to a contradiction.

Theorem 1.9.(b) Prove that if $x \neq 0$, then $x^{-1} \neq 0$

Proof Let $\boldsymbol{p: x \neq 0, q: x^{-1} \neq 0}$

we have to prove $\mathbf{p} \rightarrow \mathbf{q}$ is true statement , assume $\sim(\mathbf{p} \rightarrow \mathbf{q})$ is true statement

since $\sim(\mathrm{p} \rightarrow \mathrm{q}) = p \wedge \sim \mathrm{q}$, then $p \wedge \sim \mathrm{q}$ is true statement

$\boldsymbol{x}^{-1}.x = 0$ *but* $\boldsymbol{x}^{-1}.x = 1$ *this impels* 1= 0 which is a contradiction C!

So $\sim(\mathbf{p} \rightarrow \mathbf{q})$ is false statement .

Therefore, $\mathbf{p} \rightarrow \mathbf{q}$ is true statement.

Theorem 1.10. prove that if x is an odd number, then x^2 is odd.

Proof: Assume that x^2 is odd number. To prove x is an odd number by contradiction. Assume that $x \in E$

Then $\mathrm{x} = 2\mathrm{k}$, $\mathrm{k} \in \mathrm{Z}$

$\mathrm{x}^2 = 4\mathrm{k}^2 = 2 * 2\,\mathrm{k}^2$, $2\,\mathrm{k}^2 \in \mathrm{Z}$

$\mathrm{x}^2 \in \mathrm{E}$ this is contradiction with assumption. This implies x is an odd number.

Theorem 1.11. Prove that: If $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer **(hypothesis)**

$q: a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **(conclusion)**

Assume that the conclusion is false this means $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n}))$ is true.

But $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n}) = \sim(a \leq \sqrt{n}) \wedge \sim(b \leq \sqrt{n})$ [De Morgan's law]

$$= (a > \sqrt{n}) \wedge (b > \sqrt{n})$$

Therefore, $ab > n$ this implies to contradiction with the hypothesis.

Thus, $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is true.

Proof by mathematical induction

The principle of mathematical induction is one of the methods of mathematical proof. By using it, we can prove the validity of a statement $p(n)$for every natural number n, starting with verifying it for a base case and then proving that if it holds for $n = k$, it also holds for $n = k + 1$.

This process involves the following steps:

1. Check that $p(n)$is true for n=1 .

2. Assume that $p(n)$ is true for $n = k$ (where $k > 1$).

3. Prove that $p(n)$ is true for $n = k + 1$.

If the statement holds for n=1 and if $p(n)$ being true for $n = k$ implies it is also true for $n = k + 1$, then by induction, $p(n)$ is true for all $n \geq 1$.

Examples: 1.11. prove that the following statement is true by mathematical induction.

1) $1 + 2 + \cdots + \mathrm{n} = \frac{\mathrm{n(n+1)}}{2}$

2) $1^2 + 2^2 + \cdots + \mathrm{n}^2 = \frac{\mathrm{n(n+1)(2n+1)}}{6}$

3) $1^3 + 2^3 + \cdots + \mathrm{n}^3 = \left(\frac{\mathrm{n(n+1)}}{2}\right)^2$

Solution 1)

To prove the statement is true we use the mathematical induction

If n=1 then $1=\frac{1\times(1+1)}{2}$ is true

Assume that the statement is true when $n = k + 1$, to prove the statement is true when $n = k + 1$

$$1 + 2 + \cdots + \mathrm{k} + \mathrm{k} + 1 = \frac{\mathrm{k(k+1)}}{2} + k + 1 = \frac{\mathrm{k(k+1)}+2k+2}{2} = \frac{\mathrm{k}^2+3k+2}{2} = \frac{\mathrm{(k+1)(k+2)}}{2}$$

Therefore, the statement is true for all values of n.

Definition 1.7. (Variable) An alphabetic letter x, y, z, ... which represents a number that is either arbitrary or unknown.

Definition 1.8. (Open Sentence) A sentence is called open sentence (or propositional function), if it contains one or more variables. Open sentence is denoted by p(x), q(x), g(x)...etc.

Examples: 1.12. The following are open sentences:

$p(x)$: x is an odd number.

$q(x, y) = x + y = 5 \, , x, y \in N$

Examples: 1.13. Let the open sentence $p(x), x > 5$. What are the truth value of $p(7)\ and\ p(0)$? Which values $x \in N$ that make $p(x)$ true?

Solution: $p(7)$: 7 > 5 is a true proposition

$p(0)$: 0 > 5 is a false proposition

$p(x)$ is a true statement for $x \in \{6,7,8, \dots\}$.

Examples: 1.14. Let the open sentence $q(x,y,z)$: $x + y = z,\ x, y, z \in Z$. What are the truth values of $q(1,1,2)\ and\ q(-1,1,5)$?

Solution: $q(1,1,2)$: $1 + 1 = 2$ is a true proposition

$q(-1,1,5)$: $-1 + 1 = 5$ is a false proposition

Definition 1.9. (Solution Set or Truth Set) Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. $T_p = \{x \in A : p(x)\ is\ true\}$

Examples: 1.15. Find the solution set for each of the following open sentences:

1. $p(x), x + 2 > 7\ and\ A = N$
2. $q(x), x + 2 = 0\ and\ A = N$
3. $r(x), x + 5 > 1\ and\ A = N$

Solution: 1. $T_p = \{6,7,8, \dots\}$

2. $T_p = \emptyset$

3. $T_p = N$

Conclusion:

Logical equivalence is a crucial tool in propositional logic, aiding in the simplification and manipulation of logical expressions. By mastering the laws of logic, such as De Morgan's and the Distributive Laws, we can simplify complex expressions and better understand logical arguments.

Exercises

Exercise 1: Assume we have the following statement: "$x \leq -3\ or\ x \geq 6$". Which values of x $\in N$ that make the statement true? Which values of x that make the statement false?

Exercise 2: Assume we have the following statement: "$x > 2$ and $x < 5$". Which values of $x \in N$ that make the statement true? Which values of x that make the statement false? Discuss all the possible cases.

Exercise 3: Find the following solution sets. Also determine $p(x)$ and A for each solution set

1. $T_p = \{x \in N, -2 < x < 2\}$

2. $T_p = \{x \in Z, -1 < x < 1\}$

Exercise 4: prove that the following properties:

1. $p \to \mathrm{p} = \mathrm{T}$
2. $\sim p \to \mathrm{p} = \mathrm{p}$
3. $p \to \mathrm{T} = \mathrm{T}$
4. $T \to \mathrm{p} = \mathrm{T}$
5. $p \to \mathrm{F} = \sim p$
6. $F \to \mathrm{p} = T$
7. $p \to \mathrm{q} = \sim q \to \sim\mathrm{p}$
8. $p \to \mathrm{q} = (p \wedge \sim q) \to \sim p$
9. $p \to \mathrm{q} = (p \wedge \sim q) \to (r \wedge \sim r)$
10. $\sim(p \to \mathrm{q}) = p \wedge \sim q$

Exercise 5: Let p be a propositions. Then find the truth value of the following

1. $p \leftrightarrow \mathrm{p} = \mathrm{T}$
2. $\sim p \leftrightarrow \mathrm{p} = \mathrm{F}$
3. $p \leftrightarrow \mathrm{q} = q \leftrightarrow \mathrm{p}$
4. $p \leftrightarrow \mathrm{q} = \sim p \leftrightarrow \sim q$
5. $\sim p \leftrightarrow \mathrm{q} = p \leftrightarrow \sim q$
6. $\sim(p \leftrightarrow \mathrm{q}) = \sim p \leftrightarrow q$
7. $\sim(p \leftrightarrow \mathrm{q}) = p \leftrightarrow \sim q$

References

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