Foundations of Mathematics

Instructor: Dr. Uday Jabbar Quaez

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Department of Mathematics, College of Education, Al-Mustansiriya University

Lecture 3.: Quantifiers

Introduction

Quantifiers are special symbols used in logic and mathematics to express the quantity of elements in a given set that satisfy a certain property. They transform open sentences into statements that are either true or false.
There are two types of quantifiers:

1. Universal Quantifiers (denoted by "∀"): Denotes that a property holds for all elements in a set.
2. Existential Quantifiers (denoted by "∃"): Denotes that a property holds for at least one element in a set...

Universal Quantifiers:

The universal quantifier applies to all elements within a set. The expression "$\forall x \in A$, p(x)" translates to "for every x in the set A, the proposition p(x) is true."

Examples: 1.15. "$\forall x \in N, x > 0$"

Translation: For all natural numbers x, x is greater than zero.

This statement is true since every natural number is greater than zero.

Remark 1.12. The universal quantifier $p(x)$ on a domain A is true if and only if $T_p = A$.

Examples: 1.16. Find the truth value of the following open sentences:

1. "$\forall x \in R, x + 1 > x$"

Let $A = R$ and $p(x): x + 1 > x$

Because p(x) is true for all $x \in R$, the solution set $T_p = R$

$\Longrightarrow$ the quantification $\forall\, x \in R, x + 1 > x$ is **true.**

2. $\forall x \in N, x < 2$

Let $A = N$ *and* $p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3$, $p(3)$ is false.

$\Longrightarrow T_p \neq N$

3. $\forall x \in N, (x > 0 \text{ and } x = 0)$

The statement is **false**, there exists $x = 4 \in N$ such that $4 > 0$ *and* $4 \neq 0$.

Existential Quantifiers:

The existential quantifier applies when there is at least one element in a set that satisfies the property. The expression "there exists x in A, p(x)" means "there exists an x in set A such that p(x) is true.

Examples: 1.17. $\exists x \in N, x < 0$

Translation: There exists a natural number x that is less than zero.

This statement is false because no natural number is less than zero.

Remark 1.13. The existential quantifier $p(x)$ on a domain A is **true** if and only if $T_p \neq \emptyset$.

Examples: 1.18. Find the truth value of the following open sentences:

1. $\exists x \in R, x^2 = x$

$A = R$ *and* $p(x): x^2 = x$

$T_p = \{0, 1\}$ $\Longrightarrow$ the existential quantifier $\exists x \in R, x^2 = x$ is true

2. $\exists x \in N, 3x + 5 = 1$

$x = \frac{-4}{3} \notin N \Longrightarrow T_p = \emptyset \Longrightarrow \exists x \in N, 3x + 5 = 1$ is false

3. $\exists x \in Z, (x+1)^2 = 0$ and $x^2 - 1 = 0$

$A = Z$ *and* $p(x): (x+1)^2 = 0$ and $x^2 - 1 = 0$

$x = -1$ and $x = \mp 1$

$T_p = \{-1\} \subset Z$

$\exists x \in Z, (x+1)^2 = 0$ and $x^2 - 1 = 0$ is true

De Morgan's Law for the Existential Quantifier:

De Morgan Laws also apply to quantifiers, providing ways to transform statements involving "∀" and "∃"

- The negation of a universal quantifier: The negation of "∀" is "∃" with the negation of

the property. $\sim [\exists\, x \in A, \sim p(x)] = \forall x \in A, p(x)$

- The negation of an existential quantifier: The negation of "∃" is "∀" with the negation

of the property.

Theorem 1.14. Let p(x) be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A, p(x)] = \exists\, x \in A, \sim p(x)$
2. $\sim[\forall x \in A, \sim p(x)] = \exists\, x \in A, p(x)$
3. $\sim[\exists x \in A, p(x)] = \forall x \in A, \sim p(x)$ $(\boldsymbol{H.W.})$

Proof: 1. $\sim[\forall x \in A, p(x)] = \sim[\sim [\exists\, x \in A, \sim p(x)]]$ (De Morgan law)

$= \sim\sim[\exists\, x \in A, \sim p(x)]$

$= \exists\, x \in A, \sim p(x)$ $[\sim\sim p = p]$

2. $\sim[\forall x \in A, \sim p(x)] = \sim[\sim [\exists\, \mathrm{x} \in \mathrm{A}, \sim\sim \mathrm{p(x)}]]$ (De Morgan law)

$= \exists\, \mathrm{x} \in \mathrm{A}, \mathrm{p(x)}$ $[\sim\sim p = p]$

3. $\sim[\exists x \in A, p(x)] = \sim[\exists x \in A, \sim\sim p(x)]$ since $[\sim\sim p = p]$

$= [\forall x \in A, \sim p(x)]$ (De Morgan law)

Nested Quantifiers:

Nested quantifiers occur when two or more quantifiers are used in a single statement, each applying to different sets. For example:

"$\forall x \in A, \ \forall x \in B$, p(x, y)"

Meaning: For every x in A and every y in B, the property p(x, y) holds true.

Examples: 1.19.

1. $\forall x \in R, \forall y \in N, x^2 + y^2 \geq 0$ $(\boldsymbol{True})$

2. $\exists x \in N, \exists y \in N, \ x + y < 0$ **(False)**
3. $\exists x \in R, \forall y \in N, x + y = 0$ **(False)**

Theorem 1.15. Let x *and* y are two variables defined on the sets A and B, respectively and $p(x, y)$ an open sentence. Then:

1. $\sim[\forall x \in A, \forall y \in B, p(x, y)] = \exists x \in A, \exists y \in B, \sim p(x, y)$ (**H. W.**)
2. $\sim[\exists x \in A, \exists y \in B, p(x, y)] = \forall x \in A, \forall y \in B, \sim p(x, y)$
3. $\sim[\forall x \in A, \exists y \in B, p(x, y)] = \exists x \in A, \forall y \in B, \sim p(x, y)$ (**H. W.**)
4. $\sim[\exists x \in A, \forall y \in B, p(x, y)] = \forall x \in A, \exists y \in B, \sim p(x, y)$

Proof:

2. Take the L. H. S

$\sim[\exists x \in A, \exists y \in B, p(x, y)] = \forall x \in A, \sim [\exists y \in B, p(x, y)]$

$= \forall x \in A, \forall y \in B, \sim p(x, y)$

$=$ R. H. S

Examples: 1.20.

Find the truth values of the following statements and their negations:

1. $\forall x \in R\ (x \neq 0), \exists y \in R, xy = 1$

The statement is true because $\forall x \in R\ (x \neq 0), \exists y \in R$, $y = \frac{1}{x}, x.\frac{1}{x} = 11$

Negation:

$\sim [\forall x \in R\ (x \neq 0), \exists y \in R, xy = 1] = \exists x \in R\ (x \neq 0), \forall y \in R, xy \neq 1$

The statement is false, Let x=a$\in R$ ($a \neq 0$) then $y = \frac{1}{a} \in R$ and $xy = 1$.

2. $\exists x \in R, \exists y \in R, x^2 + y^2 \geq 0$ is true

Negation:

$\sim [\exists x \in R, \exists y \in R, x^2 + y^2 \geq 0] = \forall x \in R, \forall y \in R, x^2 + y^2 < 0$ is false

3. $\forall x \in N, \forall y \in N, x + y \in N$ (**H. W.**)
4. $\forall x \in N, \exists y \in Z, x + y \in N$ (**H. W.**)

Exercises

Exercise 1: Express the following using connective operators and/or quantifiers

1. There exists p, and there exist q such that $pq = 32$.
2. For each x, there exists y such that $x < y$.

3. Each even number is not odd number.
4. For each x, if x is natural number then x is an integer number.
5. For each natural number x, x is even number or x is odd number.

Exercise 2: Find the negation of the following sentences:

1. $\forall x, \forall y, \exists z, x + y + z = 18$
2. There exists y such for each x, $xy \leq 2$
3. $\exists x, [p(x) \to Q(x)]$

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