Foundations of Mathematics

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Chapter Two

Set Theory

Lecture 4.: Set Theory - Concepts and Applications

Introduction

In this lecture, we explore the fundamentals of set theory, a cornerstone of modern mathematics. Set theory provides the language for describing collections of objects, enabling us to study their properties and relationships. This lecture includes definitions, examples, and applications related to sets...

Definition of a Set:

A set is a collection of distinct objects, called elements or members of the set. Sets are denoted using curly braces { }. For example, A = {1, 2, 3}.

Methods for Representing Sets:

- Listing elements: All elements are listed explicitly, e.g., A = {1, 2, 3}.
- Set-builder notation: The property characterizing the set is given, e.g., $A = \{x \mid x > 0\}$.

Remark 2.1.

1. The capital letters usually used to represents sets such as A, B, C,…etc.
2. The small letters such as a, b, c, d,…etc are used to represents the members or the elements of the set.
3. Membership in a set is denoted as follows: $a \in A$ denotes that a belongs to a set A

4. Non-membership to a set is denoted as follows: a $\notin$ **A** denotes that a does not belong to a set A

Special Types of Sets

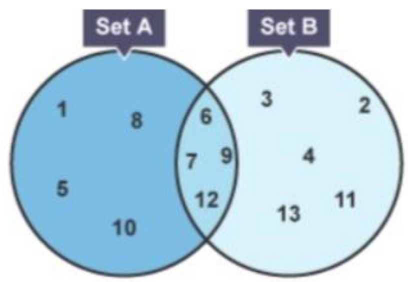
- **Empty Set**: A set with no elements, denoted by { } or empty set Φ.
- **Subsets**: A set A is a subset of B if all elements of A are also elements of B denoted by A$\subseteq B$.

$$\boldsymbol{A \subseteq B \; if \; and \; only \; if \; \forall x \in A \to x \in B}$$

$$\boldsymbol{A \nsubseteq B \; if \; and \; only \; if \; \exists x \in A \wedge x \notin B}$$

Venn Diagrams

Venn diagrams are graphical representations of sets and their relationships, such as unions and intersections.



Examples: 2.1.

1) Let $A = \{4,9\}$ and $B = \{x \in N; 1 < x < 10\}$

. Determine whether $\boldsymbol{A \subseteq B}$ $\boldsymbol{or}$ $\boldsymbol{A \nsubseteq B}$

Solution: $A \subseteq B$ since $\boldsymbol{\forall x \in A \to x \in B}$

2) Let $A = \{x \in N; x > 3\}$ and $B = \{x \in N; x^2 > 4\}$. Is $\boldsymbol{A \subseteq B}$ $\boldsymbol{or}$ $\boldsymbol{B \subseteq A}$?

Solution: $let \; x \in A \to x > 3 \to x^2 > 9 \to x^2 > 4 \to x \in B \; then$ $\boldsymbol{A \subseteq B}$

Since $3^2 > 4 \to 3 \in B$ but $3 \notin A$ $then$ $\boldsymbol{B \nsubseteq A}$.

Theorem 2.2. Let A, B and C be any sets, then

1. $\emptyset \subseteq A$
2. $A \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proof: 1. $\emptyset \subseteq A$ means $\boldsymbol{\forall x \in \emptyset \to x \in A}$

$$\boldsymbol{F \to (T \; or \; F) = T}$$

2. Let x ∈ A → x ∈ A, this means A ⊆ A

3. To Prove, If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Let $x \in A \to x \in B$ (since $A \subseteq B$)

$x \in C$ (since $B \subseteq C$)

Then, $A \subseteq C$

Definition 2.1. A set A is called a **proper** subset of B and denoted by $(A \subset B)$ if and only if $A \subseteq B$ and there exist an element $x \in B$ that is $x \notin A$.

$$A \subset B \leftrightarrow A \subseteq B \wedge \exists x \in B \wedge x \notin A$$

Examples: 2.2. $A = \{x \in N \text{ } or \text{ } x^2 - 16 = 0\}, B = \{x \in N \text{ } and \text{ } x^2 - 16 = 0\}$ Determine if $A \subset B$ or $B \subset A$.

Solution: $A = \{1, 2, 3, \dots\} \cup \{4, -4\}$ and $B = \{4\}$. It is clear that $B \subset A$ because $B \subseteq A$ and $\exists y = \{1,2,3,5,\dots\} \in A \wedge y \notin B$

Definition 2.2. Two sets A and B are equal if they both have the same elements or, equivalently, if each is contained in the other.

$$A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$$

Definition 2.3. Universal set U is the set that contains all the elements or the sets we have under discussion.

Examples: 2.3. $Let\ A = \{1,2,3\}, B = \{1,5,7\}, C = \{1,11,20\}$ then $U = N$ is possible to have a Universal set.

Definition 2.4. Family of sets is a set that have other sets as members.

Examples: 2.4. $Let\ A = \{\{1\},\{2,3\}\}, B = \{\emptyset\}, C = \{C\}$

$E = \{A \subseteq \mathrm{N}\}, M = \{\{1,2,\dots,n\}, \mathrm{n} \in \mathrm{N}\}$

Definition 2.5. The power set of a set X is the set of all possible subsets of X , including the empty set and the set itself and denoted by $P(X)$.

$$P(X) = \{A: A \subseteq X\}, \qquad A \in P(X) \Leftrightarrow A \subseteq X$$

If a set X contains n elements, the total number of subsets or elements of the power set is 2^n .

Examples: 2.5. $Let\ X = \{1,2,3\}$. Then

$$P(X) = \{A, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\},$$

Theorem 2.3. Let X and Y be two sets. Then

$$X \subseteq Y \leftrightarrow P(X) \subseteq P(Y)$$

Proof: ($\rightarrow$) suppose that $X \subseteq Y$ **to prove** $P(X) \subseteq P(Y)$

Let A $\in$ P(X) $\rightarrow$ A $\subseteq$ X (Def. of power set)

But X $\subseteq$ Y $\rightarrow$ A $\subseteq$ Y $\rightarrow$ A $\in$ P(Y) (Def. of power set)

Then P(X) $\subseteq$ P(Y)

($\leftarrow$) Suppose that P(X) $\subseteq$ P(Y) to prove X $\subseteq$ Y

Let x $\in$ X $\rightarrow$ {x} $\subseteq$ X $\rightarrow$ {x} $\in$ P(X) (Def. of power set)

But P(X) $\subseteq$ P(Y) $\rightarrow$ {x} $\in$ P(Y) $\rightarrow$ {x} $\subseteq$ Y (Def. of power set)

x $\in$ Y , then X $\subseteq$ Y

Algebra of Sets

❖ Union of Sets

Let $A, B \subseteq$ X. Then,

$A \cup B = \{x \in X; x \in A \vee x \in B\}$

i.e. $x \notin A \cup B = x \notin A \wedge x \notin B$

Examples: 2.6. let $A = \{x \in N; 2 \leq x < 7\}$ and $B = \{x \in N; 4 \leq x < 9\}$

$A \cup B = \{2,3,4,5,6,7,8\}$

$B \cup A = \{2,3,4,5,6,7,8\}$

$A \cup A = \{2,3,4,5,6\}$

Definition 2.6. If $A_1, A_2, \ldots, A_n$ are any sets in X, then

$\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$

$\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$

Examples: 2.7. **1)** Let $A_n = \{n+1, n \in N\}$

$\bigcup_{n=1}^{3} A_n = A_1 \cup A_2 \cup A_3 = \{2\} \cup \{3\} \cup \{4\} = \{2,3,4\}$

$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \dots = \{2\} \cup \{3\} \cup \dots = \{2,3,4,\dots\}$

2) Let $A_n = \{(-n,n), n \in N\}$

$\bigcup_{n=1}^{3} A_n = A_1 \cup A_2 \cup A_3 = (-1,1) \cup (-2,2) \cup (-3,3) = (-3,3)$

$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \dots = (-1,1) \cup (-2,2) \cup \dots = R$

Theorem 2.4. Let A, B *and* C be any sets in universal set X . Then

1. $A \cup \emptyset = A$ (Identity law)
2. $A \cup A = A$ (Identity law)
3. $A \cup X = X$ (Domination law)
4. $A \cup B = B \cup A$
5. $(A \cup B) \cup C = A \cup (B \cup C)$
6. $A \subseteq B \Leftrightarrow A \cup B = B$
7. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
8. $P(A) \cup P(B) \subseteq P(A \cup B)$

Proof: 1. To prove $A \cup \emptyset \subseteq A \wedge A \subseteq A \cup \emptyset$

Let $x \in A \cup \emptyset \Longrightarrow x \in A \vee x \in \emptyset$ (def. of $\cup$)

$\Longrightarrow x \in A \vee F$

$\Longrightarrow x \in A$ (p v F=p)

$\therefore A \cup \emptyset \subseteq A$ **... (1)**

Let $x \in A \Longrightarrow x \in A \vee F$ (since $p \vee F = p$)

$\Longrightarrow x \in A \vee x \in \emptyset$

$\Longrightarrow x \in A \cup \emptyset$ (def. of $\cup$)

$\therefore A \subseteq A \cup \emptyset$ **.... (2)**

From (1) & (2), $A \cup \emptyset = A$

2. To prove $A \cup A \subseteq A \wedge A \subseteq A \cup A$

Let $x \in A \cup A \Longrightarrow x \in A \vee x \in A$ (def. of ∪)

$\Longrightarrow x \in A$ (p ∨ p=p)

$\therefore A \cup A \subseteq A$ **.... (1)**

Let $x \in A \Longrightarrow x \in A \vee x \in A$ (p ∨ p=p)

$\Longrightarrow x \in A \cup A$ (def. of ∪)

$\therefore A \subseteq A \cup A$ **... (2)**

From (1) & (2), $A \cup A = A$

3. To prove $A \cup \mathrm{X} \subseteq \mathrm{X} \wedge \mathrm{X} \subseteq A \cup \mathrm{X}$

Let $x \in A \cup \mathrm{X} \Longrightarrow x \in A \vee x \in \mathrm{X}$ (def. of ∪)

$\Longrightarrow x \in \mathrm{X} \vee x \in \mathrm{X}$ (Since $A \subseteq U$)

$\Longrightarrow x \in \mathrm{X}$ (P∨P=P)

$\therefore A \cup \mathrm{X} \subseteq \mathrm{X}$ **..... (1)**

Let $x \in \mathrm{X} \Longrightarrow x \in \mathrm{X} \vee x \in A$ (Since T ∨ P=T)

$\Longrightarrow x \in A \cup \mathrm{X}$ (def. of ∪)

$\therefore \mathrm{X} \subseteq A \cup \mathrm{X}$ **...(2)**

From (1) & (2), $A \cup \mathrm{X} = \mathrm{X}$

5._ To prove $A \cup \mathrm{B} \subseteq \mathrm{B} \cup A \wedge \mathrm{B} \cup A \subseteq A \cup \mathrm{B}$

Let $x \in A \cup \mathrm{B} \Longrightarrow x \in A \vee x \in \mathrm{B}$ (def. of ∪)

$\Longrightarrow x \in \mathrm{B} \vee x \in A$ (V is commutative)

$x \in \mathrm{B} \cup A$ (def. of ∪)

$\therefore A \cup \mathrm{B} \subseteq \mathrm{B} \cup A$ **...(1)**

Similarly, show that $\mathrm{B} \cup A \subseteq A \cup \mathrm{B}$ **...(2)**

From (1) and (2), $A \cup \mathrm{B} = \mathrm{B} \cup A$

6. To prove $A \subseteq B \Leftrightarrow A \cup B = B$

($\Longrightarrow$) Let $x \in A \cup B \Longrightarrow x \in A \vee x \in B$ (def. of $\cup$)

$\Longrightarrow x \in B \vee x \in B$ (by hypo. $A \subseteq B$)

$\Longrightarrow x \in B$ (p $\vee$ p=p)

$\therefore A \cup B \subseteq B$ **...(1)**

Let $x \in B \Longrightarrow x \in B \ \vee \ x \in A$ (Since T $\vee$ p=T)

$\Longrightarrow x \in A \cup B$ (def. of $\cup$)

$\therefore B \subseteq A \cup B$ **...(2)**

From (1) and (2), $A \cup B = B$

($\Longleftarrow$) Let $x \in A \Longrightarrow x \in A \ \vee \ x \in B$ (Since T $\vee$ P=T)

$\Longrightarrow \quad x \in A \cup B$ (def. of $\cup$)

$\Longrightarrow \quad x \in B$ (by hypo. $A \cup B = B$)

$\therefore A \subseteq B$

8. Let $X \in P(A) \cup P(B)$ To prove $X \in P(A \cup B)$

$X \in P(A) \cup P(B) \Longrightarrow X \in P(A) \ \vee \ X \in P(B)$ (def. of $\cup$)

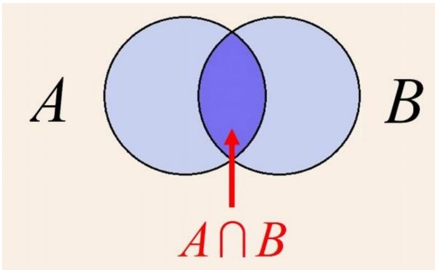
$\Longrightarrow X \subseteq A \ \vee \ X \subseteq B$ (def. of $P(A)$)

$\Longrightarrow X \subseteq A \cup B$ (def. of $\cup$)

$\Longrightarrow X \in P(A \cup B)$ (def. of $P(A \cup B)$)

$\therefore \ P(A) \cup P(B) \subseteq P(A \cup B)$

❖ Intersection of Sets



Let $A, B \subseteq$ X. Then,

$A \cap B = \{x \in X; x \in A \wedge x \in B\}$

i.e. $x \notin A \cap B = x \notin A \vee x \notin B$

Examples: 2.8. let $A = \{x \in N; 2 \leq x < 7\}$ and $B = \{x \in N; 4 \leq x < 9\}$

$A \cap B = \{4,5,6\}$

$B \cap B = \{4,5,6\}$

$A \cap A = \{2,3,4,5,6\}$

Definition 2.7. If $A_1, A_2, \ldots, A_n$ are any sets in X, then

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$

$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots$

Examples: 2.7. **1)** Let $A_n = \{n+1, n \in N\}$

$\bigcap_{n=1}^{3} A_n = A_1 \cap A_2 \cap A_3 = \{2\} \cap \{3\} \cap \{4\} = \emptyset$

$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \ldots = \emptyset$

2) Let $A_n = \{(-n, n), n \in N\}$

$\bigcap_{n=1}^{3} A_n = A_1 \cap A_2 \cap A_3 = (-1,1) \cap (-2,2) \cap (-3,3) = (-1,1)$

$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \ldots = (-1,1) \cap (-2,2) \cap \ldots = (-1,1)$

Theorem 2.5. Let A, B *and* C be any sets in universal set X . Then

1. $A \cap \emptyset = \emptyset$ (Domination law)
2. $A \cap A = A$ (Idempotent law)
3. $A \cap X = A$ (Identity law)
4. $A \cap B = B \cap A$
5. $(A \cap B) \cap C = A \cap (B \cap C)$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \subseteq B \Leftrightarrow A \cap B = A$
9. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
10. $P(A) \cap P(B) = P(A \cap B)$

Proof: 2. *For each* $x \in A \Leftrightarrow x \in A \wedge x \in A$ $(p \wedge p = p)$

$x \in A \Leftrightarrow x \in A \cap B$ (def. of $\cap$)

$\therefore A \cap A = A$

3. To prove $A \cap X \subseteq A \wedge A \subseteq A \cap X$

Let $x \in A \cap X \Longrightarrow x \in A \wedge x \in X$ (def. of $\cap$)

$\Longrightarrow x \in A$

$\therefore A \cap X \subseteq A$ **...(1)**

Let $x \in A \Longrightarrow x \in A \wedge x \in X$ [since $A \subseteq X$]

$\Longrightarrow x \in A \cap X$ **...(2)**

From (1) and (2), $A \cap X = A$

5. Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$ (def. of $\cap$)

$\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ (def. of $\cap$)

$\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ ($\wedge$ is assoc.)

$\Leftrightarrow x \in A \wedge x \in B \cap C$ (def. of $\cap$)

$\Leftrightarrow x \in A \cap (B \cap C)$ (def. of $\cap$)

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

7. Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in B \cup C$ (def. of $\cap$)

$\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ (def. of $\cup$)

$\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ (distribute $\wedge$ on v)

$\Leftrightarrow x \in A \cap B \vee x \in A \cap C$ (def. of $\cap$)

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ (def. of $\cup$)

8. To prove $A \subseteq B \Leftrightarrow A \cap B = A$

($\Longrightarrow$) Let $x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of $\cap$)

$\Longrightarrow x \in A$

$\therefore A \cap B \subseteq A$ **....(1)**

Let $x \in A \Longrightarrow x \in A \wedge x \in B$ [$A \subseteq$ B]

$\Longrightarrow x \in A \cap B$

$\Longrightarrow A \subseteq A \cap B$ **....(2)**

From (1) and (2), $A \cap B = A$

($\Longleftarrow$) Let $A \cap B = A$, to prove $A \subseteq B$

Let $x \in A \Longrightarrow x \in A \cap B$ (by hyp.)

$x \in A \cap B \Longrightarrow x \in A \wedge x \in$ B

$\therefore x \in \mathrm{B} \Longrightarrow A \subseteq B$

Exercises

Exercise 1: Let $A_n = \left\{\left(-\frac{1}{n}, \frac{1}{n}\right), n \in N\right\}$. Find $\bigcup_{n=3}^{6} A_n$, $\bigcup_{n=1}^{\infty} A_n$ **and** $\bigcap_{n=1}^{\infty} A_n$

Exercise 2: Let $A_n = \{(n, n+1), n \in N\}$. Find $\bigcup_{n=1}^{3} A_n$ *and* $\bigcup_{n=1}^{\infty} A_n$

Exercise 3: Let $A = \{x \in Z: -2 \leq x \leq 10\}$, $B = \{x \in Z \text{ v } x^2 + 9 = 0\}$. Determine if $A \subset B$ *or* $B \subset A$.

Exercise 4: Let $A = \{-2, 3\}$, $B = \{x \in Z: x^3 - x^2 - 6x = 0\}$. Determine whether $A \subseteq B$ or $B \subseteq A$?

Exercise 5: Let $A = \{x \in \mathrm{N}: x \geq 4\}$ and $B = \{x \in \mathrm{N} : x < 9\}$. Determine whether $A \subseteq B$ or$B \subseteq A$?

Exercise 6: Let $X = \{A_n; A_n = \{n-2, n-1, n\}; n \in N\}$. Find $\bigcap_{n=1}^{\infty} A_n$,$\bigcup_{n=1}^{\infty} A_n$

Exercise 7: Let $X = \{A_n; A_n = \{n^3 + 1\}; n \in Z\}$. Find

1. $\bigcap_{n=-3}^{0} A_n$,
2. $\bigcap_{n=1}^{\infty} A_n$
3. $P(\bigcup_{n=1}^{3} A_n)$

References

1. Smith, P. (2003). *Introduction to Mathematical Logic*. Cambridge University Press. ISBN: 9780521008044.
2. Enderton, H. B. (1977). *Elements of set theory*. Academic press.
3. Shoenfield, J. R. (2000). *Mathematical Logic*. A K Peters. ISBN: 9781568811352.
4. Levy, A. (2012). *Basic set theory*. Courier Corporation..
5. Fraenkel, A. (1973). *Foundations of Set Theory*. Amsterdam: North-Holland.