Foundations of Mathematics

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Lecture 5.: Complement and Difference of Sets

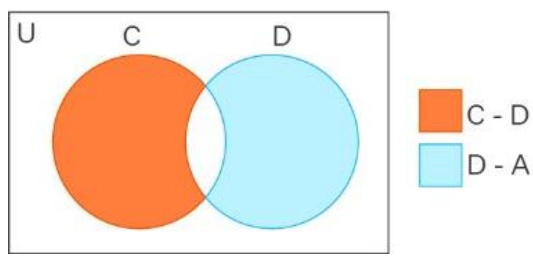
Introduction

In mathematics, the complement and difference of sets are fundamental operations in set theory. These concepts allow us to analyze relationships between sets and their elements. Understanding these operations helps in various mathematical applications, including logic, probability, and computer science.

Definition of Difference of Sets:

For two sets A and B, the difference A and B denoted by $A - B$ is a set of all elements that belong to A but not to B.

Formally: $A - B = \{x \in A \land x \notin B\}$



Examples: 2.10. Let $A = N \ and \ B = \{1, 2, 3\}$. Then

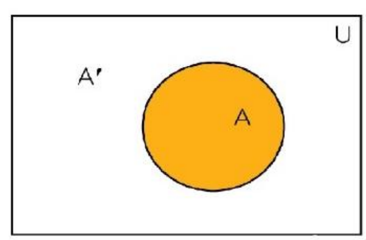
$A - B = \{4,5,6, \dots\}$

$A - \emptyset = A$

$A - A = \emptyset$

Definition 2.7. The complement of a set A, within a universal set X, includes all the elements in X that are not in A. It is denoted by A^c or $X - A$ and is defined as:

$$A^c = \{x \in X \land x \notin A\}$$



Examples: 2.11. 1) Let $X = N$ and $A = \{1, 2, 3\}$. Then

$A^c = \{4,5,6, \dots\}$

2) Let $X = N$ and $A = N$. Then $A^c = \emptyset$

3) Let $X = R$ and $A = \emptyset$. Then $A^c = R$

4) Let $X = N, A = \{1, 2\}, B = \{2,3,4\}$. Then

$A^c = N - A = \{3,4,5, \dots\}$

$B^c = N - B = \{1,5,6,7, \dots\}$

$(A \cap B)^c = N - (A \cap B) = N - \{2\} = \{1,3,4,5, \dots\}$

$(A \cup B)^c = N - (A \cup B) = N - \{1,2,3,4\} = \{5,6,7, \dots\}$

$A^c \cap B^c = \{5,6,7, \dots\}$

$A^c \cup B^c = \{1,3,4,5, \dots\}$

Theorem 2.7. Let A, B *and* C be any sets in universal set X. Then

1. $\emptyset^c = X$
2. $A \cap A^c = \emptyset$
3. $A \cup A^c = X$
4. $(A^c)^c = A$
5. $(A \cup B)^c = A^c \cap B^c$
6. $(A \cap B)^c = A^c \cup B^c$
7. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$
8. $A \subseteq B \Leftrightarrow A \cap B^c = \emptyset$

Proof: 1. Assume that $\emptyset^c \neq X$, $\exists x, x \in X \wedge x \notin \emptyset^c$

$x \in X \wedge x \in \emptyset$ this is C!

$\therefore \emptyset^c = X$

2. Assume that $A \cap A^c \neq \emptyset$

$\exists x \in X, x \in A \cap A^c \Longrightarrow x \in A \wedge x \in A^c$ $(Def. of \cap)$

$\Longrightarrow x \in A \wedge x \notin A$ $(Def. of Complement\ set)$

this is C! $\Longrightarrow \therefore A \cap A^c = \emptyset$

3. Assume that $A \cup A^c \neq X$

$\exists x \in X \wedge x \notin A \cup A^c$

$x \notin A \cup A^c \Longrightarrow x \notin A \wedge x \notin A^c$ $(Def. of \cup \text{ and } De-morgan\ Lows)$

This is contradiction with the definition of complement set.

$\therefore A \cup A^c = X$

4. $(A^c)^c = A$ to prove $A \subseteq (A^c)^c$ and $(A^c)^c \subseteq A$

Let $x \in A \Longrightarrow x \notin A^c$ $(Def. of Complement\ set)$

$\Longrightarrow x \in (A^c)^c \Longrightarrow \therefore A \subseteq (A^c)^c$ ... (1)

Let $x \in (A^c)^c \Longrightarrow x \notin A^c$ $(Def. of Complement\ set)$

$\Longrightarrow x \in A \Longrightarrow \therefore (A^c)^c \subseteq A$ ... (2)

From (1) and (2), we have $(A^c)^c = A$

5. $(A \cup B)^c = A^c \cap B^c$ to prove $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in (A \cup B)^c \Longrightarrow x \notin A \cup B$ $(Def. of Complement\ set)$

$\Longrightarrow x \notin A \wedge x \notin B$ $(Def\ of\ \cup)$

$\Longrightarrow x \in A^c \wedge x \in B^c$ $(Def. of Complement\ set)$

$\Longrightarrow x \in A^c \cap B^c$ $(Def. of\ \cap)$

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$ ... (1)

Similarity, we prove $A^c \cap B^c \subseteq (A \cup B)^c$ ... (2)

From (1) and (2), we have $(A \cup B)^c = A^c \cap B^c$

6. $(A \cap B)^c = A^c \cup B^c$

to prove $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$

Let $x \in (A \cap B)^c \Longrightarrow x \notin A \cap B$ $(Def. of Complement\ set)$

$\Longrightarrow$ x $\notin$ A $\vee$ x $\notin$ B $(Def. of \cap and\ De-morgan\ Lows)$

$\Longrightarrow$ x $\in A^c \vee$ x $\in B^c$ $(Def. of Complement\ set)$

$\Longrightarrow$ x $\in A^c \cup B^c$ $(Def. of\ \cup)$

$\therefore (A \cap B)^c \subseteq A^c \cup B^c$ ... (1)

Similarity, we prove $A^c \cup B^c \subseteq (A \cap B)^c$ ... (2)

From (1) and (2), we have $(A \cap B)^c = A^c \cup B^c$

7. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$ $(\Longrightarrow)$

Suppose that $A \subseteq B$ **, to prove** $B^c \subseteq A^c$

Let x $\in B^c \Longrightarrow$ x $\notin$ B $(Def. of Complement\ set)$

Since $A \subseteq B \Longrightarrow$ x $\notin$ A $(Def. of subset)$

$\therefore$ x $\in A^c$ $(Def. of Complement\ set)$

$\therefore B^c \subseteq A^c$

$(\Leftarrow)$ Suppose that $B^c \subseteq A^c$ **, to prove** $A \subseteq B$

Let x $\in A \Longrightarrow$ x $\notin A^c$ $(Def. of Complement\ set)$

Since $B^c \subseteq A^c \Longrightarrow$ x $\notin B^c$ $(Def. of subset)$

$\therefore$ x $\in B$ $(Def. of Complement\ set)$

$\therefore A \subseteq B$

Theorem 2.8. Let A, B and C be any sets in universal set X. Then

1. $A - A = \emptyset$ and $A - X = \emptyset$
2. $A - \emptyset = A$ and $\emptyset - A = \emptyset$
3. $A - B \subseteq A$
4. $A \subseteq B \Leftrightarrow \mathrm{A} - \mathrm{B} = \emptyset$
5. $(A - B) \cap B = \emptyset$
6. $A \cap B = \emptyset \Leftrightarrow (A - B = A) \wedge (B - A = B)$
7. $A - (B \cup C) = (A - B) \cap (A - C)$

8. $A-(B \cap C) = (A-B) \cup (A-C)$

9. $A-A^c = A$ $and\ A^c - A = A^c$

***Proof:* 1.**To prove $A-A=\emptyset$

Suppose that $A-A \neq \emptyset \Longrightarrow \exists x \in X, x \in A-A$

$\Longrightarrow x \in A \wedge x \notin A$ $(Def. of difference\ set)$

This implies to contradiction. $\therefore A-A=\emptyset$

3. To prove $A-B \subseteq A$

Let $x \in A-B \Longrightarrow x \in A \wedge x \notin B$ $(Def. of difference\ set)$

$\therefore x \in A$ $(p \wedge T = p)$

4. ***To prove*** $A \subseteq B \Leftrightarrow A-B=\emptyset$

$(\Longrightarrow)$Suppose that $A \subseteq B$ to prove $A-B=\emptyset$

Assume that $A-B \neq \emptyset \Longrightarrow \exists x \in X, x \in A-B$

$\Longrightarrow x \in A \wedge x \notin B$ $(Def. of difference\ set)$

But, $A \subseteq B$? $x \in A \Longrightarrow x \in B$. *This implies to contradiction* ***since*** $x \in B \wedge x \notin B$

$\therefore A-B=\emptyset$

$(\Leftarrow)$ Suppose that $A-B=\emptyset$ **, to prove** $A \subseteq B$

Assume that $A \nsubseteq B \Longrightarrow \exists x \in X, x \in A \wedge x \notin B$ $(Def. of\ subset)$

$\therefore x \in A-B$ $(Def. of difference\ set)$

$\Longrightarrow A-B \neq \emptyset$ This implies to contradiction. $\therefore A \subseteq B$

5. Assume that $(A-B) \cap B \neq \emptyset \Longrightarrow \exists x, x \in (A-B) \cap B$

$\Longrightarrow x \in A-B \wedge x \in B$ $(Def. of\ \cap)$

$\Longrightarrow (x \in A \wedge x \notin B) \wedge x \in B$ $(Def. of difference\ set)$

$\Longrightarrow x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is associative)

$\Longrightarrow x \in A \wedge F$ this implies that the proposition is false $(p \wedge F = F)$

$\Longrightarrow \therefore (A - B) \cap B = \emptyset$

6. $(\Longrightarrow)$ Let $A \cap B = \emptyset$, **to prove** $A - B = A \wedge B - A = B$

Let $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$ $(Def. of difference\ set)$

$\Leftrightarrow x \in A \wedge T$ (since $A \cap B = \emptyset$)

$\Leftrightarrow x \in A$ $(p \wedge T = p)$

$\therefore A - B = A$

In the Similar method, we prove that $B - A = B$Type equation here.

$(\Leftarrow)$Let $A - B = A \wedge B - A = B$, **to prove** $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset \Longrightarrow \exists x \in X, x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (Def. of $\cap$)

$\Longrightarrow \exists x, x \in A - B \wedge x \in B$ $(A - B = A)$

$\Longrightarrow \exists x, x \in A - B \wedge x \in B$

$\Longrightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B$

$\Longrightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is associative)

$\Longrightarrow \exists x, x \in A \wedge F = F$ $(p \wedge F = F)$

$\therefore A \cap B = \emptyset$

8. To prove $A - (B \cap C) = (A - B) \cup (A - C)$

Let $x \in A - (B \cap C) \Leftrightarrow x \in A \wedge x \notin (B \cap C)$ $(Def. of difference\ set)$

$\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C)$ $(Def. of \cap\ and\ De - morgan\ Lows)$

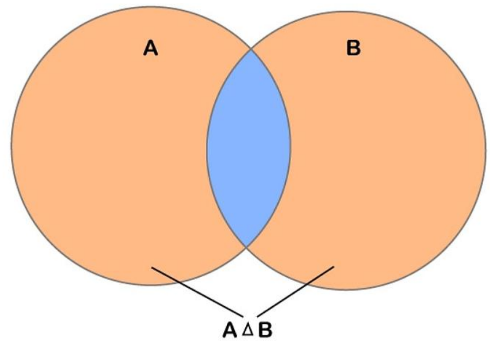
$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$ $(\wedge\ distributives\ on\ \vee)$

$\Leftrightarrow x \in (A - B) \vee x \in (A - C)$ $(Def. of difference\ set)$

$\Leftrightarrow x \in (A - B) \cup (A - C)$ $(Def\ of\ \cup)$

Definition of Symmetric Difference Sets:

The symmetric difference between two sets A and B is denoted by A∆B and is defined as follows:



$$A \Delta B = (A - B) \cup (B - A)$$

$$= (A \cup B) - (A \cap B)$$

Examples: 2.12.

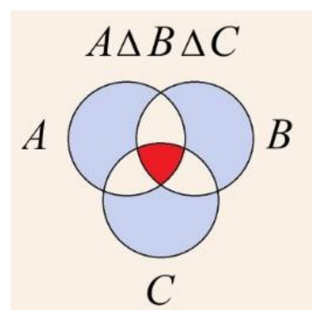
Let $A = N$ and $B = \{1, 2, 3\}$. Then

$A\Delta B = (A \cup B) - (A \cap B) = N - \{1, 2, 3\} = \{4,5,6, \dots\}$

$A\Delta\emptyset = (A \cup \emptyset) - (A \cap \emptyset) = A$

$A\Delta A = \emptyset$

Theorem 2.9. Let A, B and C be any sets in universal set X. Then



1. $A\Delta A = \emptyset$
2. $A\Delta B = B\Delta A$
3. $A \Delta B = \emptyset \Leftrightarrow A = B$
4. $A \Delta (B \Delta C) = (A\Delta B)\Delta C$
5. $A \cap (B \Delta C) = (A \cap B)\Delta(A \cap C)$

Exercises

Exercise 1: Prove or disprove the following

1. $A \cup B = A \cap B \Leftrightarrow A = B$

2. $A - (B \cup C) = \emptyset \Rightarrow A \subseteq (B \cup C)$

3. $P(A - B) = P(A) - P(B)$

Exercise 2: Prove or disprove the following

1. $P(A \cap B) \subseteq P(A) \cap P(B)$

2. $(A - B)^c = B \cup A^c$

Exercise 3: Let $A = \{x \in N: 1 \leq x < 5\}$, $B = \{2, 4, 6\}$ and $C = N$.

$Find$ $A - (B \cup C)$, $A - (B \cap C)$

Exercise 4: If $A = \emptyset$ then find $P(A)$ and $P(A^c \cap A)$

Exercise 5: Let $A \cap B = \emptyset$. Is $A - B = B - A$, Explain that?

Exercise 6: Let $X = \{A_n; \ A_n = \left(\frac{-1}{n}, \frac{1}{n}\right),; \ n \in N\}$. Find $\bigcap_{n=1}^{4}(A_1 - A_n)$,$\bigcup_{n=1}^{4}(A_1 - A_n)$

Exercise 7: Let $A = \{x \in E: -8 \leq x < 9\}$ and $B = \{1, 2, 4, 6\}$ Find $A \Delta B$.

References

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