Foundations of Mathematics

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Academic Year: 2025-2026
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Chapter Three

Relations

Lecture 6.: Cartesian Product

Introduction

The Cartesian product is a fundamental concept used to build many ideas in mathematics, engineering, and programming. It is defined as the operation that produces a set of ordered pairs, where the first element of each pair comes from the first set, and the second element comes from the second set.

Ordered pair:

Ordered pair is a mathematical concept used to represent two elements arranged in a specific order where the order of elements is important. It is written in the form (a, b) where a is the first element and b is the second element.

Properties of Ordered Pairs:

Let a, b, c and d be elements. Then:

1. $(\mathrm{a}, \mathrm{b}) = (\mathrm{c}, \mathrm{d}) \Leftrightarrow \mathrm{a} = \mathrm{c} \wedge \mathrm{b} = \mathrm{d}$
2. $(\mathrm{a}, \mathrm{b}) = (\mathrm{b}, \mathrm{a})$ if $a \neq b$

Cartesian Product:

If A and B are two sets, their Cartesian product is denoted by $\mathrm{A} \times \mathrm{B}$ and is defined as :

$$\mathrm{A} \times \mathrm{B} = \{(\mathrm{a}, \mathrm{b}); \mathrm{a} \in \mathrm{A} \wedge \mathrm{b} \in \mathrm{B}\}$$

The size of Cartesian product is calculated as: if A has n elements and B has m elements then $A \times B$ has $m \times n$ elements.

Examples: 3.1. let $A = \{1,2\}$ and $B = \{3,4\}$

$$A \times B = \{(1,3),(1,4),(2,3),(2,4), \}$$

$$A \times A = \{(1,1),(1,2),(2,1),(2,2), \}$$

$$B \times A = \{(3,1),(3,2),(4,1),(4,2), \}$$

$$A \times \emptyset = \emptyset$$

Theorem 3.1. Let A, B, D and C be any sets in universal set X. Then

1. If $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$
2. $A \times B = B \times A \Leftrightarrow A = B$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
5. $A \times (B - C) = (A \times B) - (A \times C)$
6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
7. If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

Proof:1. Suppose that $A \times B \neq \emptyset$**,** this means $\exists\ (a,b) \in A \times B \Longrightarrow$
$a \in A \wedge b \in B$ (Def. of $A \times B$)
$\Longrightarrow A \neq \emptyset \wedge B \neq \emptyset$, this implies to C!
$\therefore A \times B = \emptyset$

2. $(\Longrightarrow)$ Suppose that $A \times B = B \times A$ to prove $A = B$
For each $(x,y) \in A \times B \Longrightarrow x \in A \wedge y \in B$ (Def. of $A \times B$)
$(x,y) \in B \times A \Longrightarrow x \in B \wedge y \in A$ $(A \times B = B \times A)$
$\Longrightarrow A \subseteq B \wedge B \subseteq A$
$\Longrightarrow A = B$

$(\Leftarrow)$ Suppose that $A = B$ to prove $A \times B = B \times A$
Let $(x,y) \in A \times B \Leftrightarrow x \in A \wedge y \in B$ (Def. of $A \times B$)
$\Leftrightarrow x \in B \wedge y \in A$ $(A = B)$
$\Leftrightarrow (x,y) \in B \times A$
$\therefore A \times B = B \times A$

3. To prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let $(x,y) \in A \times (B \cap C) \Leftrightarrow x \in A \wedge y \in (B \cap C)$ (Def. of $A \times B$)

$\Leftrightarrow x \in A \wedge (y \in B \wedge y \in C)$ (Def. of $\cap$)

$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)$ ($\wedge$ distributives on $\wedge$)

$\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in A \times C$ (Def. of Cartesian Product)

$\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C)$ (Def. of $\cap$)

$\therefore A \times (B \cap C) = (A \times B) \cap (A \times C)$

5. $A \times (B - C) = (A \times B) - (A \times C)$

Let $(x,y) \in A \times (B - C) \Leftrightarrow x \in A \wedge y \in (B - C)$ (Def. of Cartesian Product)

$\Leftrightarrow x \in A \wedge (y \in B \wedge y \notin C)$ (Def. of difference)

$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$

($\wedge$ distributives on $\wedge$)

$\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \notin A \times C$

$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$ (dist. $\wedge$ on $\wedge$)

$\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \notin A \times C$ (Def. of Cartesian Product

$\Leftrightarrow (x,y) \in (A \times B) - (A \times C)$ (Def. of difference)

$\therefore A \times (B - C) = (A \times B) - (A \times C)$

6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Let $(x,y) \in (A \times B) \cap (C \times D)$

$\Leftrightarrow (x,y) \in A \times B \wedge (x,y) \in C \times D$ (Def. of $\cap$)

$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D)$ (Def. of Cartesian Product

$\Leftrightarrow (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D)$ ($\wedge$ commut. and associ.)

$\Leftrightarrow x \in (A \cap C) \wedge y \in (B \cap D)$ (Def. of $\cap$)

$\Leftrightarrow (x,y) \in (A \cap C) \times (B \cap D)$ (Def. of Cartesian Product

Definition 3.1. If $A, A_1, A_2, \dots, A_n$ are sets in X, then

$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n); x_1 \in A_1 \wedge x_2 \in A_2 \wedge \dots \wedge x_n \in A_n\}$

$A^n = A \times A \times \dots \times A$

Examples: 3.2. 1) let $A = \{1,2\}, B = \{1\}$ and $C = \{3,4\}$. Then

$A \times B \times C = \{(1,1,3), (1,1,4), (2,1,3), (2,1,4)\}$

$A^2 = A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

2) let R be a set of real numbers. Then

$R^2 = R \times R$

$R^3 = R \times R \times R$

Exercises

Exercise 1: Prove or disprove the following

1. $A \times B = A \times C \Leftrightarrow B = C$

2. $A \times (B \times C) = (A \times B) \times C$

Exercise 2: Prove or disprove the following

1. $P(A \times B) \subseteq P(A) \times P(B)$

2. $P(A \cup B) = P(A) \cup P(B)$

Exercise 3: Let $A = \{x \in N: 2 \leq x < 4\}$, $B = \{2, 3, 6\}$ and $C = N$.

Find $A \times (B \cup C)$, $A \times (B \cap C)$

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