Foundations of Mathematics

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Lecture 7.: Relations

Introduction

Relations are a fundamental concept in mathematics that play a significant role in organizing and connecting different entities. They are widely used in various applications such as databases, set theory, and data analysis. In this lectures, we will delve into the concept of relations, explore their types and properties, and understand how they can be applied in different fields.

Relation:

A relation $\mathcal{R}$ between two sets A and B is a subset of Cartesian product $A \times B$. In other words: $\mathcal{R} \subseteq A \times B$. The following notations are used for relation:

$(a, b) \in \mathcal{R} \Leftrightarrow a\mathcal{R}b \; or \; a \sim b$

$(a, b) \notin \mathcal{R} \Leftrightarrow a\mathcal{R}b \; or \; a \nsim b$

The notations $R, S, W, \ldots$ are used to express the relations. If $R \subseteq A \times A$ then R is called a relation on A**.**

Examples: 3.3. If $A = \{1,2\} \; and \; B = \{3,4\}$ then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Define the following relations:

$R = \{(1, 3), (1, 4)\}$

$S = \{(1, 3)\}$

$W = \{(2, 3), (2, 4)\}$

Remark 3.2. $\emptyset$ and $A \times B$ are relations from $A \times B$ such that $\emptyset$ is the smallest relation and $A \times B$ is the largest relation.

Methods for Representing Relations:

- Listing elements: All elements are listed explicitly, e.g., $R = \{(1,3),(1,4)\}$.

- Set-builder notation: The property characterizing the relation is given, e.g.,

$W = \{(x, y),\ y = 2x, x, y \in N\}$

Let x and y be integers. Then we say that x divides y (x factor y) denoted by $x|y$, if there exists an integer k such that $xk = y$.

$$x|y \Leftrightarrow \exists k \in Z, y = kx$$

$$x \text{ not divides } y \Leftrightarrow \forall k \in Z, y \neq kx$$

Examples: 3.4.

- $3|6 \Leftrightarrow \exists 2 \in Z, 6 = 2 \times 3$
- $3|12 \Leftrightarrow \exists 4 \in Z, 12 = 4 \times 3$
- $-5|15 \Leftrightarrow \exists -3 \in Z, 15 = -3 \times -5$
- $A = \{x \in Z: 0 \leq x \leq 4\}$. If $R = \{(a,b) \in A \times A: a|b\}$ then

  $R = \{(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,2),(2,4),(3,0),(4,0),(4,4)\}$

Remark 3.3. Let R_1 and R_2 are two relations from A to B, then $R_1 \cap R_2$, $R_1 \cup R_2$ and $R_1 - R_2$ are also relations from A to B.

Examples: 3.5. let $A = \{x \in Z; -3 < x < 3\}$. The relations R_1 and R_2 are defined as $R_1 = \{(-1,1),(0,0)\}$ and $R_2 = \{(-2,2),(0,0),(1,1)\}$, find

$R_1 \cap R_2 = \{(0,0)\}$

$R_1 \cup R_2 = \{(-1,1),(0,0),(1,1),(-2,2)\}$

$R_1 - R_2 = \{(-1,1)\}$

Definition 3.2. Let R be a relation from A to B. Then R^{-1} is also relation from B **to** A and said to be the inverse of R.

$R^{-1} = \{(b,a);\ (a,b) \in R\}$

Clearly, $(R^{-1})^{-1} = R$

Examples: 3.6. Let $A = \{x \in Z; -3 < x < 3\}$ and $R = \{(x, y); y = 2x, x, y \in A\}$

Write the elements of the relations R and R^{-1}?

$R = \{(-1, -2), (0,0), (1,2)\}$

$R^{-1} = \{(-2, -1), (0,0), (2,1)\}$

Definition 3.3. Let R be a relation from A to B. Then the domain and range of R are defined as follows, respectively

$\text{dom } R = \{x \in A; \exists y \in B: (x, y) \in R\}$

$\text{range } R = \{y \in B; \exists x \in A: (x, y) \in R\}$

Clearly, dom R $\subseteq$ A and range R $\subseteq$ B

Examples: 3.7. Let $R = \{(x, y) \in N \times N; y = 2x\}$. Find dom R and range R?

$R = \{(1,2), (2,4), (3,6), \dots\}$

$\text{dom } R = \{1,2,3, \dots\} = N$

$\text{range } R = \{2,4,6, \dots\} = E$

Lemma 3.4. : Let R be a relation on $A \times B$ then:

1. $\text{dom } R = \text{range } R^{-1}$
2. $\text{range } R = \text{dom } R^{-1}$

proof: 1. Let $x \in dom\ R \Leftrightarrow \exists y \in B: (x, y) \in R$ *(Def. of dom R)*

$\Leftrightarrow \exists y \in B: (y, x) \in R^{-1}$ *(Def. of R^{-1})*

$\Leftrightarrow x \in \text{range } R^{-1}$ *(Def. of range R)*

In the similar method to prove 2.

Properties of Relations:

1) Reflexive relation: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every $a \in A$

2) Symmetric Relation: A relation R on a set A is called **symmetric** if $(a, b) \in R$ then $(b, a) \in R$.

3) Transitive Relation: A relation R on a set A is called **transitive** if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Examples: 3.8. Let $A = \{1,2,3\}$. What type the following relations:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$R_2 = \{(1,1)\}$

$R_3 = \{(1,1), (1,2), (2,2), (3,3)\}$

$R_4 = \emptyset$

$R_5 = \{(1,1), (1,3), (3,2), (2,2), (3,3)\}$

Solution: R_1 is reflexive on A because

$(1,1) \in R_1, (2,2) \in R_1, (3,3) \in R_1 \Longrightarrow (a, a) \in R_1\ \forall a \in A$

R_1 is symmetric on A because

$(1,2) \in R_1 \Longrightarrow (2,1) \in R_1 \Longrightarrow \forall (a, b) \in R_1$ then $(b, a) \in R_1$.

R_1 is transitive on A because

$(1,2) \in R_1 \wedge (2,1) \in R_1 \Longrightarrow (1,1) \in R_1 \Longrightarrow \forall (a, b) \in R_1 \wedge (b, c) \in R_1$ then $(a, c) \in R_1$.

....

R_2 is not reflexive on A because

$2 \in A$ but $(2, 2) \notin R_2$

R_2 is symmetric on A because $\forall (a, b) \in R_2$ then $(b, a) \in R_2$.

R_2 is transitive on A because $\forall (a, b) \in R_2 \wedge (b, c) \in R_2$ then $(a, c) \in R_2$.

....

R_3 is reflexive on A because

$(1,1) \in R_3, (2,2) \in R_3, (3,3) \in R_3 \Longrightarrow (a, a) \in R_3\ \forall a \in A$

R_3 is not symmetric on A because $(1, 2) \in R_3$ but $(2, 1) \notin R_3$.

R_3 is transitive on A because $\forall(\mathrm{a,b}) \in R_3 \wedge (b,c) \in R_3$ then $(\mathrm{a,c}) \in R_3$.

....

R_4 is not reflexive on A because $2 \in A$ but $(2,2) \notin R_4$

R_4 is symmetric on A.

R_4 is transitive on A.

....

R_5 is reflexive on A.

R_5 is not symmetric on A because $(1,3) \in R_5$ but $(3,1) \notin R_5$.

R_5 is not transitive on A because $(1,3) \in R_5 \wedge (3,2) \in R_5$ but $(1,2) \notin R_5$

Examples: 3.9. Let $A = \{-1,0,1,2\}$. What type the following relations:

$R_1 = \{(x,y); x = y\}$

$R_2 = \{(x,y); x \leq y\}$

$R_3 = \{(x,y); x < y\}$

Solution: R_1 is reflexive on A because $(x,x) \in R_1 \ \forall x \in A$

R_1 is symmetric on A because $\forall(\mathrm{x,y}) \in R_1$ then $(\mathrm{y,x}) \in R_1$.

R_1 is transitive on A because $\forall(\mathrm{x,y}) \in R_1 \wedge (y,z) \in R_1$ then $(\mathrm{x,z}) \in R_1$.

....

R_2 is reflexive on A because $\forall x \in A \Longrightarrow x \leq x \Longrightarrow (x,x) \in R_2$

R_2 is not symmetric on A because $(-1,0) \in R_2$ but $(0,-1) \notin R_2$.

R_2 is transitive on A because

$if\ (\mathrm{x,y}) \in R_2 \Longrightarrow x \leq y \wedge (y,z) \in R_2 \Longrightarrow y \leq z$ then $x \leq z \Longrightarrow (\mathrm{x,z}) \in R_2$.

....

R_3 is not reflexive on A because $-1 \in A \Longrightarrow -1 < -1\ is\ F \Longrightarrow (-1,-1) \notin R_3$

R_3 is not symmetric on A because $(-1,0) \in R_3$ but $(0,-1) \notin R_3$.

R_3 is transitive on A because

$if\ (x,y) \in R_3 \Longrightarrow x < y \wedge (y,z) \in R_3 \Longrightarrow y < z$ then $x < z \Longrightarrow (x,z) \in R_3$.

Examples: 3.10. Let $A = Z$ and $\mathcal{R}$ be a relation on A such that $\mathcal{R} = \{(x,y); x = y\ or\ x = -y\}$

1. Is $\mathcal{R}$ is a reflexive?
2. Is $\mathcal{R}$ is a symmetric?

Solution:1. $\mathcal{R}$ is reflexive on A because $\forall x \in A \Longrightarrow x = x \Longrightarrow (x,x) \in \mathcal{R}$

2. $\mathcal{R}$ is symmetric on A because $\forall (x,y) \in \mathcal{R} \Longrightarrow x = y\ or\ x = -y$

$if\ x = y \Longrightarrow (y,x) \in \mathcal{R} \quad or \quad x = -y \Longrightarrow -x = y \Longrightarrow (y,x) \in \mathcal{R}$

Examples: 3.11. Let $A = N$ and $\mathcal{R}$ be a relation on A such that

$\mathcal{R} = \{(x,y); x|y\}$

1. Is $\mathcal{R}$ is a reflexive?
2. Is $\mathcal{R}$ is a symmetric?

Solution:1. $\mathcal{R}$ is reflexive on A because $\forall x \in A \Longrightarrow x|x \Longrightarrow (x,x) \in \mathcal{R}$ (choose k=1 then x=1.x)

2. $\mathcal{R}$ is not symmetric on A because $1|2 \Longrightarrow (1,2) \in \mathcal{R}$ but 2 *not divides* $1 \Longrightarrow (2,1) \in \mathcal{R}$ (if $2|1 \Longrightarrow\Longleftrightarrow \exists k \in Z,\ 1 = k \times 2 \Longrightarrow k = \frac{1}{2}\ \ C!$)

Definition 3.4. Let A be a set. The identity relation on A denoted by I_A is defined as follows:

$I_A = \{(a,b) \in A \times A;\ a = b\}$

I_A satisfies the properties reflexive, symmetric and transitive.

Examples: 3.12. Let $A = \{1,2,3\}$, then

$I_A = \{(1,1),(2,2),(3,3)\}$

Examples: 3.13. Let $A = Z$ and define $a \sim b \Longleftrightarrow ab \geq 0\ \ \forall a,b \in Z$

Is the relation reflexive? Symmetric? Transitive?

Solution: Reflexive since for any $a \in Z$, $a.a = a^2 \geq 0 \Longrightarrow a \sim a$ then the relation is reflexive.

Symmetric since $\forall a, b \in Z$, $a \sim b \Leftrightarrow ab \geq 0$

$$\Leftrightarrow ba \geq 0 \Leftrightarrow b \sim a$$

then the relation is symmetric.

Transitive since $\forall a, b, c \in Z$, $a \sim b \wedge b \sim c \Leftrightarrow ab \geq 0 \wedge bc \geq 0$

$$\Leftrightarrow abbc \geq 0$$

$$\Leftrightarrow ab^2c \geq 0$$

But, $b^2 \geq 0$ then $ac \geq 0 \Leftrightarrow a \sim c$ then the relation is transitive.

Examples: 3.14. Let $A = R$ and define $a\,\mathcal{R}\,b \Leftrightarrow a - b > 0 \;\; \forall a, b \in \mathcal{R}$

Is $\mathcal{R}$ reflexive? Symmetric? Transitive?

Solution: Reflexive since for any $a \in R$, $a - a = 0 \Longrightarrow (a, a) \notin \mathcal{R}$ then the relation is not reflexive.

Symmetric since $\forall a, b \in R$, $(a, b) \in \mathcal{R} \Leftrightarrow a - b > 0$

$$\Leftrightarrow b - a < 0 \Leftrightarrow (b, a) \notin \mathcal{R}$$

then the relation is not symmetric.

Transitive since $\forall a, b, c \in R$,

$(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \Leftrightarrow a - b > 0 \wedge b - c > 0$

$$\Leftrightarrow a - b + b - c > 0 \Leftrightarrow a - c > 0 \Leftrightarrow (a, c) \in \mathcal{R}$$

then the relation is transitive.

Theorem 3.5. : A relation $\mathcal{R}$ on a set A is symmetric iff $\mathcal{R} = \mathcal{R}^{-1}$.

Proof: $(\Longrightarrow)$ Suppose $\mathcal{R}$ is symmetric T. P. $\mathcal{R} = \mathcal{R}^{-1}$

Let $(a, b) \in \mathcal{R} \Leftrightarrow (b, a) \in \mathcal{R}$ ($\mathcal{R}$ is symmetric)

$$\Leftrightarrow (a,b) \in \mathcal{R}^{-1} \quad \text{(Def. of } \mathcal{R}^{-1}\text{)}$$

$$\mathcal{R} = \mathcal{R}^{-1}$$

$(\Leftarrow)$ Suppose $\mathcal{R} = \mathcal{R}^{-1}\boldsymbol{\mathcal{R}}$ T. P. is symmetric

Let $(a,b) \in \mathcal{R} \Leftrightarrow (a,b) \in \mathcal{R}^{-1}$ $(\mathcal{R} = \mathcal{R}^{-1})$

$$\Leftrightarrow (b,a) \in (\mathcal{R}^{-1})^{-1} = \mathcal{R}$$

$\mathcal{R}$ is symmetric

Anti-Symmetric Relation:

A relation $\boldsymbol{\mathcal{R}}$ on a set A is called anti-symmetric if for any $a, b \in A$ then $\boldsymbol{\mathcal{R}}$ satisfies the statements

$$a{\sim}b \wedge b{\sim}a \Longrightarrow a = b$$

Examples: 3.15. let $A = \{1,2,3\}$. Are the following relations on a set A anti symmetric?

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ not anti symmetric

$R_2 = \{(1,1)\}$ anti symmetric

$R_3 = \{(1,1),(1,2),(2,2),(3,3)\}$

$R_4 = \emptyset$

$R_5 = \{(1,1),(1,3),(3,2),(2,2),(3,3)\}$

Examples: 3.16. let $A = Z$. Are the following relations on a set A reflexive ? Symmetric? Anti-symmetric?

1) $a{\sim}b \Leftrightarrow a = b + 1$

<u>Reflexive</u> since for any $a \in Z$, $a{\sim}a \Leftrightarrow a \neq a + 1$ then the relation is not reflexive.

<u>Symmetric</u> since $\forall a, b \in Z$, $a{\sim}b \Leftrightarrow a = b + 1$ but $b \neq a + 1$ then $(b,a) \notin R$ then the relation is not symmetric.

<u>Anti-symmetric</u> Since $\forall a, b \in Z$ If $a{\sim}b \wedge b{\sim}a \Longrightarrow a = b + 1 \wedge b = a + 1 \Longrightarrow a = b$ then the relation is not anti-symmetric.

2) $a \sim b \Leftrightarrow a|b$

Reflexive since for any a ∈ Z , $a \sim a \Leftrightarrow a|a$ $(a = 1 \times a)$ then the relation is reflexive.

Symmetric since ∀a, b ∈ Z , $a \sim b \Leftrightarrow a|b$ but not nessary $b|a$

For example 2|4 but 4|2 then the relation is not symmetric.

Anti-symmetric Since ∀a, b ∈ Z If $a \sim b \wedge b \sim a \Longrightarrow a|b \wedge b|a \Longrightarrow \exists k_1, k_2 \in Z$ such that $b = k_1 a$ and $a = k_2 b \Longrightarrow k_1, k_2 = 1$

$\Longrightarrow$ ether $k_1 = k_2 = 1$ or $k_1 = k_2 = -1$

$\Longrightarrow a = b$ or $a = -b$ then the relation is not anti-symmetric.

3) $a \sim b \Leftrightarrow a - b = 2k \ , k \in Z$ (H.W.)

4) $a \sim b \Leftrightarrow a = 2b$

Theorem 3.6. : A relation $\mathcal{R}$ on a set A is anti-symmetric iff $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$.

Proof: $(\Longrightarrow)$ Suppose that $\mathcal{R}$ is anti-symmetric T.P. $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$

Let $(a, b) \in \mathcal{R} \cap \mathcal{R}^{-1} \Longrightarrow (a, b) \in \mathcal{R} \wedge (a, b) \in \mathcal{R}^{-1}$ (Def. of ∩)

$\Longrightarrow (a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R}$ (Def. of $\mathcal{R}^{-1}$)

$\Longrightarrow a = b$ ($\mathcal{R}$ is anti-symmetric)

$\Longrightarrow (a, b) \in I_A$

$\Longrightarrow (a, b) \in R \wedge (b, a) \in R$

$\Longrightarrow a = b$ ($\mathcal{R}$ is anti symmetric)

$\Longrightarrow (a, b) \in I_A$

$(\Longleftarrow)$ Suppose that $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_A$ T.P. $\mathcal{R}$ is anti-symmetric

Let $(a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R} \Longrightarrow (a, b) \in \mathcal{R} \wedge (a, b) \in \mathcal{R}^{-1}$ (Def. of $\mathcal{R}^{-1}$)

$\Longrightarrow (a, b) \in \mathcal{R} \cap \mathcal{R}^{-1}$ (Def. of ∩)

$\Longrightarrow (a, b) \in I_A$ (By hyp.)

$$\Longrightarrow a = b$$

$\mathcal{R}$ is anti-symmetric.

Equivalence Relation:

A relation $\mathcal{R}$ on a set A is called equivalence relation if and only if it satisfies reflexive, symmetric and transitive properties.

Examples: 3.16. let $A = Z$. Are the following relations on a set A Equivalence?

1) $a \sim b \Leftrightarrow a + b = 2k , k \in Z$

2) $a \sim b \Leftrightarrow a|b$

Solution: (H.W.)

Partition of a Set:

A collection of subsets $\{A_i : i \in J \subseteq N\}$ of A is called partition of A iff it satisfies the following conditions :

1. $\mathrm{A_i} \neq \emptyset \quad \forall \mathrm{i} \in \mathrm{J}$

2. $\mathrm{A_i} \cap \mathrm{A_j} = \emptyset \quad \forall \mathrm{i} \neq \mathrm{j}$

3. $\bigcup_{\mathrm{i} \in \mathrm{J}} \mathrm{A_i} = \mathrm{A}$

Examples: 3.17. 1) let $A = Z$ and E and O are even and odd numbers respectively,

1. $\mathrm{E} \neq \emptyset$ and $\mathrm{O} \neq \emptyset$

2. $\mathrm{E} \cap \mathrm{O} = \emptyset$

3. $\mathrm{E} \cup \mathrm{O} = \mathrm{Z}$

then $\mathrm{p} = \{\mathrm{E}, \mathrm{O}\}$ is a partition of A

2) $A = (0,4)$ find Partition of A?

Let $\mathrm{A_1} = (0,1), \mathrm{A_2} = (1,2), \mathrm{A_3} = (2,3), \mathrm{A_3} = (3,4), \mathrm{A_5} = \{1,2,3\}$. Then

1. $A_i \neq \emptyset \quad \forall i \in J = \{1,2,3,4,5\}$

2. $A_i \cap A_j = \emptyset \quad \forall i \neq j$

3. $\bigcup_{i \in J} A_i = A$

3) find the another partition of A in 2) (H.W.)

4) $A = [-1,5)$ find three different Partitions of A?

Exercises

Exercise 1: Let $A = Z$ and define $a\,\mathcal{R}\,b \iff |a| = |b|, \forall a, b \in Z$

Is $\mathcal{R}$ reflexive? Symmetric? Transitive?

Exercise 2: Let $A = Z$ and define $a\,\mathcal{R}\,b \iff a = 1, \forall a \in Z$

$\mathcal{R} = \{(1, b); b \in Z\}$. Is $\mathcal{R}$ reflexive? Symmetric? Transitive?

Exercise 3: Let $R = \{(a, b) \in Z \times Z: b = 1 - a\}$. find dom R and range R.

Exercise 4: Let $A = R$ and define $a \sim b \iff ab < 0 \quad \forall a, b \in R$

Is the relation reflexive? Symmetric? Transitive?

Exercise 5: Let $A = Z$ and $\mathcal{R}$ be a relation on A such that $\mathcal{R} = \{(x, y); x = y \text{ and } x = -y\}$

1. Is $\mathcal{R}$ is a reflexive?
2. Is $\mathcal{R}$ is a symmetric?
3. Is $\mathcal{R}$ is a transitive?

Exercise 6: Let $A = Z$ and define $a\,\mathcal{R}\,b \iff |a| = |b|, \forall a, b \in Z$

Is $\mathcal{R}$ symmetric? Anti-symmetric?

Exercise 7: Let $A = Z$ and define $a\,\mathcal{R}\,b \iff a = 1, \forall a \in Z$

$\mathcal{R} = \{(1, b); b \in Z\}$. Is $\mathcal{R}$ equivalence?

Exercise 8: Let $R = \{(a, b) \in Z \times Z: b = 1 - a\}$. Is R symmetric ? Anti-symmetric? .

Exercise 9: Let $A = R$ and define $a \sim b \iff ab \leq 0 \quad \forall a, b \in R$

Is the relation equivalence?

Exercise 10: Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 5k, k \in Z\}$. Show that R is an equivalence relation?

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