Foundations of Mathematics

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Lecture 8.: Equivalence Classes on the Integer Set and Partially, Totally Ordered Relations

Introduction

In mathematics, Equivalence classes are used to group elements of a set according to a shared property, where all elements in the same class are equivalent under a given relation. For example, the set of integers can be divided into groups based on their remainders when divided by a fixed number. This concept forms a fundamental idea in studying relations and their applications in abstract mathematics.

Additionally, the concept of order plays a fundamental role in understanding the structure and organization of elements within a set. Two important types of order that arise frequently are partially ordered relations and totally ordered relations. Among the different types of order, partially ordered relations provide a versatile framework for analyzing situations where not all elements are directly comparable. In this lecture, we will explore the formal definition of partially ordered relations, examine examples and counterexamples, and discuss their significance in mathematical structures such as posets (partially ordered sets). By the end, you will gain a deeper understanding of how partial orders are used to represent and analyze hierarchical relationships in various mathematical and real-world contexts.

Introduction and Simple Examples:

Let us consider a simple example to introduce the concept of **equivalence classes**. Suppose we have the relation R on the set of integers $\mathbb{Z}$ defined as:

$$aRb \Leftarrow a - b \text{ is divisible by } 3$$

That is, two integers a and b are related if their difference is a multiple of 3.

For instance:

$$4R1 \quad \text{because} \quad 4-1=3 \text{ is divisible by } 3$$

$$7\,R\,10 \quad \text{because} \quad 7-10=-3 \text{ is divisible by } 3$$

This relation divides the integers into groups (or classes) where every integer in the same group has the same remainder when divided by 3:

$$[0]=\{\dots,-6,-3,0,3,6,9,\dots\}$$

$$[1]=\{\dots,-5,-2,1,4,7,10,\dots\}$$

$$[2]=\{\dots,-4,-1,2,5,8,11,\dots\}$$

These groups are called **equivalence classes**.

Equivalence Class:

Let R be an equivalence relation on a set A. The **equivalence class** of an element $a \in A$ under R is defined as:

$$[a]=\{x \in A|xRa\}$$

That is, $[a]$ contains all elements related to a under R.

Examples: 3.18.

For the relation $aRb \Leftarrow a-b$ is divisible by 3, the equivalence class [4] is:

$$[4]=\{x \in \mathbb{Z}|x-4 \text{ is divisible by } 3\}=\{\dots,-2,1,4,7,10,\dots\}$$

Thus, $[4]=[1]$ since 4 and 1 are related.

Theorem 3.7. Let R be an equivalence relation on A. Then for any $a, b \in A$,

$$aRb \Leftrightarrow [a]=[b]$$

Proof: $(\Rightarrow)$

Suppose $[a]=[b]$. Since $a \in [a]$, it follows that $a \in [b]$, which means aRb.

$(\Leftarrow)$

Suppose aRb. To show $[a]=[b]$, take any $x \in [a]$. Then xRa. Since aRb and R is transitive, we get xRb, so $x \in [b]$. Hence $[a] \subseteq [b]$. Similarly, by symmetry and transitivity, $[b] \subseteq [a]$. Therefore, $[a]=[b]$.

Theorem 3.8 If R is an equivalence relation on A, then for any $a, b \in A$,

$$[a]=[b] \text{ or } [a]\cap[b]=\emptyset$$

Proof: Assume that $[a]\cap[b]\neq\emptyset$. Then there exists some $x \in A$ such that $x \in [a]$ and $x \in [b]$, i.e., xRa and xRb. By the symmetry of R, aRx, and by transitivity, aRb. By Theorem 3.7, $[a]=[b]$. Thus, any two equivalence classes are either equal or disjoint.

Theorem 3.9 Every equivalence relation on a nonempty set A partitions A into disjoint equivalence classes.

Proof: Let R be an equivalence relation on A. Define the set of all equivalence classes:

$$\mathcal{P} = \{[a]; a \in A\}$$

Then, Each $a \in A$ belongs to some equivalence class $[a]$, so $\bigcup_{a\in A} [a] = A$.
By Theorem 3.8, any two classes are either equal or disjoint. Hence, the set of equivalence classes forms a partition of A.

Examples: 3.19. The relation $aRb \Leftarrow a \equiv b(\text{mod}3)$ on $\mathbb{Z}$ partitions the integers into three equivalence classes:

$$[0], [1], [2]$$

Thus, we can represent the partition as:

$$\mathbb{Z} = [0] \cup [1] \cup [2]$$

and the classes are pairwise disjoint.

Partially Ordered Relation

A relation $\mathcal{R}$ on a set A is called a Partially Ordered Relation (POR) if it satisfies the properties

- Reflexive
- Antisymmetric
- Transitive

The pair $(\mathcal{R}, A)$ is called partially ordered set.

Examples: 3.20. let $A = \{1,2,3\}$. Are the following relations on a set A Partially Ordered Relation (POR)?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$ is not anti symmetric , therefore the pair (R_1, A) is not (POR).

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$R_2 = \{(1,1)\}$ is not refelexive therefore is not (POR).

$R_3 = \{(1,1), (1,2), (2,2), (3,3)\}$ is Reflexive, Antisymmetric and Transitive this implies the pair (R_3, A) is not (POR).

(POR).

$R_4 = \emptyset$ is not refelexive therefore is not (POR).

$R_5 = \{(1,1),(1,3),(3,2),(2,2),(3,3)\}$ is not transitive therefore the pair (R_5, A) is not (POR).

Examples: 3.21. let $A = Z$. Are the following relations on a set A (POR)?

1) $a \sim b \Leftrightarrow a = b + 1$

Since for any a ∈ Z , $a \sim a \Leftrightarrow a \neq a + 1$ then the relation is not reflexive also the pair $(\mathcal{R}, A)$ not (POR)

2) $a \sim b \Leftrightarrow a|b$

Reflexive since for any a ∈ Z , $a \sim a \Leftrightarrow a|a$ $(a = 1 \times a)$ then the relation is reflexive.

Anti-symmetric Since ∀a, b ∈ Z If $a \sim b \wedge b \sim a \Longrightarrow a|b \wedge b|a \Longrightarrow \exists k_1, k_2 \in Z \text{ such that } b = k_1 a \text{ and } a = k_2 b \Longrightarrow k_1, k_2 = 1$

$\Longrightarrow ether\ k_1 = k_2 = 1\ or\ k_1 = k_2 = -1$

$\Longrightarrow a = b\ or\ a = -b$ then the relation is not anti-symmetric. Hence, the relation is not the pair $(\mathcal{R}, A)$ *is* not (POR)

Examples: 3.22. Show that $(Z, \geq)$ Partially Ordered Relation (POR)?

Solution: Suppose that R is a relation on a set A that defined as

$R = \{(a, b); a, b \in Z , a \geq b \}$. To show that the pair (R, Z) *is* (POR)

Reflexive since for any a ∈ Z , $aRa \Leftrightarrow a = a$ then the relation is reflexive.

Anti-symmetric Since ∀a, b ∈ Z If $aRb \wedge bRa \Longrightarrow a \geq b \wedge b \geq a \Longrightarrow a = b$ therefore, R is antisymmetric.

Transitive ∀a, b ∈ Z If $aRb \wedge bRc \Longrightarrow a \geq b \wedge b \geq c \Longrightarrow a \geq c$ this means that the relation R is transitive

Consequently, $(Z, \geq)$ is Partially Ordered Relation (POR).

Remark: 1. The largest relation $A \times A$ on a set A is not Partially Ordered Relation (POR). (Check)

2. The smallest relation $\emptyset$ on a nonempty set is not Partially Ordered Relation (POR). (Check)

Definition (3.5): Let $(\mathcal{R}, A)$ be a partially ordered set. Then, the elements $a, b \in A$ are called comparable with respect to $\mathcal{R}$ if $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$.

Examples: 3.23. Let $A = \{1,2,3\}$ *and* let

$R = \{(1,1), (2,2), (3,3), (1,2), (3,1), (3,2)\}$ be a (POR) on A. Find the comparable elements in A with respect to R

$1R\,1 \implies 1,1$ *are comparable*

$2R2 \implies 2,2$ *are comparable*

$3R3 \implies 3,3$ *are comparable*

$1R2 \implies 1,2$ *are comparable*

$3R1 \implies 1,3$ *are comparable*

$2R3 \implies 2,3$ *are comparable*

Totally Ordered Relation:

A relation $\mathcal{R}$ on a set A is called a totally Ordered Relation (TOR) if it satisfies the properties

a. $(\mathcal{R}, A)$ is a partially ordered set.
b. a *and* b are comparable , $\forall a, b \in A$

Examples: 3.24. Let $A = \{1,2,3\}$ and let

$R = \{(1,1), (2,2), (3,3), (1,2), (3,1), (3,2)\}$. Is the relation R on a set A (TOR)?

Solution:

<u>Reflexive</u> since for any $a \in A$, $aRa \iff$ then the relation is reflexive.

Anti-symmetric Since $\forall a, b \in A$ If $aRb \wedge bRa \Longrightarrow a = b$ therefore, R is antisymmetric.

Transitive $\forall a, b \in Z$ If $aRb \wedge bRc \Longrightarrow aRc$ this means that the relation R is transitive.

Then, (R, A) is a partially ordered set. Also, from Example 3.21 for each two elements in A are comparable, therefore R on a set A is a totally ordered relation (TOR).

Examples: 3.25. Show that $(Z, \geq)$ Partially Ordered Relation (TOR)?(H.W.)

Exercises

Exercise 1: Let $X = \{1,2,3\}$ and $R = \{(A, B) \in P(X) \times P(X): A \subseteq B\}$

Show that R is a partially ordered relation on $P(X)$

Exercise 2: Give an example of a (POR) that is not (TOR) ?

Exercise 3: Show that $(N, \geq)$ Totally Ordered Relation (TOR)? Discuss

$(R, \geq)$?

Exercise 4: Let $A = Z$ and $R = \{(a, b) \in Z \times Z: a - b = 5k, k \in Z\}$. Is R is a totally ordered relation (TOR)?

References

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