∴ The method of Triangular Decomposition (LU) ∴

The square matrix "A" can be decomposed in to two Triangular matrix:-

L : A lower Triangular matrix.
U: An upper Triangular matrix.
Thus $\qquad A = LU$

To obtain A From the linear equations system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow A = LU$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{13} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{3} \end{bmatrix}$$

∴ The method of Triangular Decomposition (LU) ∴

To Find U and L following The five steps:-

(1) $U_{11} = a_{11}$, $U_{12} = a_{12}$, $U_{13} = a_{13}$

(2) $L_{21} U_{11} = a_{21} \longrightarrow \boxed{L_{21}} = \frac{a_{21}}{U_{11}}$

$L_{31} U_{11} = a_{31} \longrightarrow \boxed{L_{31}} = \frac{a_{31}}{U_{11}}$

(3) $L_{21} U_{12} + U_{22} = a_{22} \longrightarrow \boxed{U_{22}} = a_{22} - L_{21} U_{12}$

$L_{21} U_{13} + U_{23} = a_{23} \longrightarrow \boxed{U_{23}} = a_{23} - L_{21} U_{13}$

(4) $L_{31} U_{12} + L_{32} U_{22} = a_{32}$

$\rightarrow \boxed{L_{32}} = \frac{a_{32} - L_{31} U_{12}}{U_{22}}$

(5) $L_{31} U_{13} + L_{32} U_{23} + U_{33} = a_{33}$

$\boxed{U_{33}} = a_{33} - L_{31} U_{13} - L_{32} U_{23}$

To Find $\vec{L} \vec{Y} = \vec{b}$ To Find $\vec{Y}$

To Find $\vec{U} \vec{X} = \vec{Y}$ To Find $\vec{X}$

∴ The method of Triangular Decomposition ∴

Ex:- By using LU method. Find the solution of the following systems?

$$2x_1 + x_2 + 3x_3 = 1$$
$$6x_1 + 2x_2 + 7x_3 = 0$$
$$4x_1 + 8x_2 + 2x_3 = 2$$

Solution:-

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 2 & 7 \\ 4 & 8 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{L_{21}} & 1 & 0 \\ \boxed{L_{31}} & \boxed{L_{32}} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

To Find $\boxed{U} \rightarrow \quad U_{11} = 2 \quad , U_{12} = 1 \quad , U_{13} = 3$

(2) To Find $L_{21} \rightarrow \quad L_{21} = \dfrac{a_{21}}{U_{11}} = \dfrac{6}{2} = 3$

$$L_{31} = \frac{a_{31}}{U_{11}} = \frac{4}{2} = 2$$

∴ من الخطوة الاولى اوجدنا السطر الاول من U
والعمود الاول من L

(3) To Find U_{22}, U_{23}

$$U_{22} = a_{22} - L_{21} U_{12} = 2 - 3 = -1$$
$$U_{23} = a_{23} - L_{21} U_{13} = 7 - 3(3) = -2$$

اوجدنا السطر الثاني من U $\rightarrow$ $\therefore U_{22} = -1 \quad , U_{23} = -2$

(4) $L_{32} = \dfrac{a_{32} - L_{31} U_{12}}{U_{22}} = \dfrac{8-2}{-1} = -6$

(5) $U_{33} = a_{33} - L_{31} U_{13} - L_{32} U_{23}$

$= 2 - 2(3) - (-6)(-2) = -16$

$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -6 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -16 \end{bmatrix}$

Now To Find $\vec{Y}$ From $\vec{L}\vec{Y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\rightarrow y_1 = 1$.

$3y_1 + y_2 = 0 \rightarrow y_2 = -3$

$2y_1 - 6y_2 + y_3 = 2 \rightarrow y_3 = -18$

$$\rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -18 \end{bmatrix}$$

$\rightarrow$ To Find $\vec{x}$ From $\vec{U}\vec{x} = \vec{Y}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -18 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1.562 \\ 0.75 \\ 1.125 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$