[1]

* Sufficient Statistics *

likelihood Function :-

The likelihood Function of a r.s. $X_1, \dots, X_n$ of size (n) From a distribution with pdf $f(x)$ with parameters $\theta_1, \theta_2, \dots, \theta_n$ is defined to be the joint pdf of the n. r.v's $X_1, X_2, \dots, X_n$ which considered as a function of θ's and denoted by $L(\bar{\theta}, \bar{x})$:-

$$L(\bar{\theta}, \bar{x}) = L(\theta_1, \theta_2, \dots, \theta_n; x_1, x_2, \dots, x_n)$$

$$= f(\bar{x}, \bar{\theta}) = \prod_{i=1}^{n} f(x_i)$$

$$\pi(x_1, \dots, x_n / t) = \frac{L(\theta; x_1, x_2, \dots, x_n)}{g(t; \theta)}$$

Conditional Distribution

if T is a S.S. for θ, then

$$= \pi(x_1, \dots, x_n; t)$$

* Conditional Distribution *

let X and y be any Two r.v. with joint density function $f(x,y)$, The conditional probability density function of x given $Y=y$ is defined by :-

$$f(x | Y=y) = \frac{f(x,y)}{f_2(y)}, \quad f_2(y) > 0$$

Sufficient Statistics

Sufficient Statistics can solved by Two ways :-

By Definition

By Using Neyman's Factorization theorem.

1) By using Definition:-

let $X_1, X_2, \dots, X_n$ be a r.s. from dis. $F(x,\theta)$ and let $T = u(X_1, \dots, X_n)$ be a statistics whose pdf is $g(t,\theta)$:-

* A sufficient Statistics, T is a statistics which contains all the information for the estimation of θ.

* A sufficient Statistics in not unique.

* The conditional distribution of sample r.v's given the value of τ of T is defined as:-

$$\pi(x_1, \dots, x_n / t) = \frac{F(x_1, \dots x_n, t; \theta)}{g(t,\theta)}$$

$$= \frac{L(\theta; x_1, \dots x_n)}{g(t,\theta)}$$

Notes :-

1) $\prod_{i=1}^{n} c = (c)^n$.

2) $\prod_{i=1}^{n} e^{x} = e^{\sum_{i=1}^{n} x_i}$.

3) $\prod_{i=1}^{n} a^{x} = a^{\sum_{i=1}^{n} x_i}$.

4) $\prod_{i=1}^{n} e^{a} = e^{na}$.

5) $\sum_{i=1}^{n} m = mn$.

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Ex1) let $X_i, i=1,\dots,n$ be a r.s. From Poisson dist. with Parameter θ
Show that $T=\sum_{i=1}^{n} x_i$ is S.S. for θ?

Solution) By using Definition.

$$\pi(x_1,\dots,x_n/t)=\frac{L(\theta;x_1,\dots,x_n)}{g(t;\theta)}$$

$$\because X_i \sim P(\theta) \rightarrow f(x,\theta)=\begin{cases}\frac{e^{-\theta}\cdot\theta^{x}}{x!} & , x=0,1,\dots \\ 0 & , o.w.\end{cases}$$

where $L(\theta;X)=f(x_1,\theta).f(x_2,\theta)\dots f(x_n,\theta)$

$$=\frac{e^{-\theta}\cdot\theta^{x_1}}{x_1!}\cdot\frac{e^{-\theta}\cdot\theta^{x_2}}{x_2!}\cdots\frac{e^{-\theta}\cdot\theta^{x_n}}{x_n!}$$

$$=\prod_{i=1}^{n}\left[\frac{e^{-\theta}\cdot\theta^{x_i}}{x_i!}\right]=\frac{e^{-\theta n}\cdot\theta^{\sum_{i=1}^{n}x_i}}{(\prod_{i=1}^{n}x_i)!}$$

Since $X\sim P(\theta)\rightarrow T=\sum_{i=1}^{n}X_i \sim P(n\theta)$

From properties
(الفصل الأول)

$$\Rightarrow g(t;\theta)=\frac{e^{-n\theta}\cdot(n\theta)^{t}}{t!}$$

$$\Rightarrow \pi(x_1,x_2,\dots,x_n/t)=\frac{\dfrac{e^{-n\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{(\prod_{i=1}^{n}x_i)!}}{\dfrac{e^{-n\theta}\cdot(n\theta)^{t}}{t!}}$$

$$=\frac{\dfrac{1}{(\prod_{i=1}^{n}x_i)!}}{\dfrac{(n)^{t}}{t!}}$$

$\therefore$ T is a S.S. for θ.

Ex2) let $X_1, X_2, \ldots, X_n$ be a r.s. from $Ber(\theta)$, Show that $T = \sum_{i=1}^{n} x_i$ is a S.S. for θ?

Solution) $\because x_i \sim Ber(\theta)$

$$\rightarrow f(x;\theta) = \begin{cases} \theta^{x}(1-\theta)^{1-x} & , x = 0, 1, \ldots, n \\ 0 & , \text{o.w} \end{cases}$$

$$\prod f(x_i;\theta) = f(x_1,\theta) \cdots f(x_n,\theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1} \cdot \ldots \cdot \theta^{x_n}(1-\theta)^{1-x_n}$$

$$= \prod_{i=1}^{n} \left(\theta^{x_i}(1-\theta)^{1-x_i}\right)$$

$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} x_i} = \theta^{t} \cdot (1-\theta)^{t-n}$$

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$$\because x_i \sim Ber(\theta) \rightarrow y = \sum_{i=1}^{n} x_i \sim Bin(n,\theta)$$

$$\rightarrow T = \sum_{i=1}^{n} x_i \sim Bin(n,\theta)$$

$$\therefore g(t;\theta) = C_t^n \theta^t (1-\theta)^{n-t}$$

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$$S.S. \Rightarrow \pi(x_1, \ldots, x_n / t) = \frac{\theta^t \cdot (1-\theta)^{n-t}}{C_t^n \theta^t (1-\theta)^{n-t}} = \frac{1}{C_t^n}$$

$$\therefore T = \sum_{i=1}^{n} x_i \text{ is S.S. for } \theta.$$

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* Second way: Neyman's Factorization theorem *

Let $X_1, \dots X_n$ be ar.s. from a distribution has pdf $f(x;\theta)$ and $T = t(x_1, \dots, x_n)$ be a statistics whose pdf $g(t;\theta)$

$$\Rightarrow L(\theta) = g(t;\theta) \cdot h(x_1, \dots, x_n).$$

where g and h are non-negative functions.

Ex1) let $X_i \sim N(\theta, \sigma^2)$, $i = 1, 2, \dots, n$, Show that $T = \sum_{i=1}^{n} x_i$ is s.s. for θ?

Solution:- By using Neyman's Factorization theorem.

$$L(\theta;X) = f(x_1;\theta) \cdot f(x_2,\theta) \dots f(x_n,\theta)$$

$$\because X_i \sim N(\theta, \sigma^2) \rightarrow f(x;\theta) = \begin{cases} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma^2}\right)^2}, & \text{exc} \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore L(x;\theta) = f(x_1,\theta) \dots f(x_n,\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}} \right)$$

$$= \left(\frac{1}{\sigma \cdot \sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} (x_i - \theta)^2}$$

$$L(x;\theta) = g(t;\theta) \cdot h(x_1, \dots, x_n)$$

$$\rightarrow g(t;\theta) = e^{-\frac{1}{2\sigma^2} \cdot \left(-2\theta \sum_{i=1}^{n} x_i + n\theta^2\right)}$$

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$$h(x_1, \ldots, x_n) = e^{-\frac{1}{2\sigma}\cdot\left(\sum_{i=1}^{n} x_i\right)}$$

$$\therefore T = \sum_{i=1}^{n} x_i \text{ is S.S. for } \theta.$$

Ex2) let $X_1, X_2, \ldots, X_n$ be a r.s. from Poisson dist. with parameter θ, show that $T = \sum_{i=1}^{n} X_i$ is S.S. for θ? [by using Neyman's th.]

Solution) By using Neyman's Factorization theorem.

$$\because X_i \sim P(\theta) \Rightarrow f(x;\theta) = \begin{cases} \dfrac{e^{-\theta}\cdot\theta^{x}}{x_i} \\ 0, \text{ o.w.} \end{cases}$$

$$L(x;\theta) = g(t;\theta)\cdot h(x_1, \ldots, x_n)$$

$$L(x;\theta) = \prod_{i=1}^{n} f(x_i, \theta) \leftarrow$$

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$$\prod_{i=1}^{n} f(x_i, \theta) = \frac{e^{-n\theta}\cdot\theta^{\sum_{i=1}^{n} x_i}}{\left(\prod_{i=1}^{n} x_i\right)!}$$

$$L(x;\theta) \Rightarrow g(t;\theta) = e^{-n\theta}\cdot\theta^{\sum_{i=1}^{n} x_i}$$

$$h(x_1, \ldots, x_n) = \frac{1}{\left(\prod_{i=1}^{n} x_i\right)!}$$

$$\therefore T = \sum_{i=1}^{n} x_i \text{ is S.S. for } \theta.$$