∴ Order Statistics ∴

P.d.f of the largest order statistics:-

- if $X_1, X_2, \dots, X_n$ be a r.s. of size n from a population with continuous P.d.f $f(x)$, then the P.d.f of the largest order statistic Y_n is given as:-

$$g(y_n) = n\left(F_x(y_n)\right)^{n-1} \cdot f_x(y_n)$$

Example:- Suppose X_1, X_2 are i.i.d (identically independent distributed) Uniform $(\theta, 1)$ random variables, find the density function of y_2.

Sol:- $\because g(y_2)$ and $n = 2$

$\therefore$ p.d.f of the largest O.S.

$$\rightarrow g(y_n) = n\left(F_x(y_n)\right)^{n-1} \cdot f_x(y_n)$$

$$\because n = 2 \rightarrow g(y_2) = 2\left(F_x(y_2)\right)^{2-1} \cdot f_x(y_2)$$

$\because X \sim Uni(\theta, 1)$

$$\therefore f(x) = \begin{cases} \dfrac{1}{1-\theta}, & \theta < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\rightarrow f_x(y_2) = \frac{1}{1-\theta}, \quad F(x) = \int_{\theta}^{x} f(u)\,du$$

∴ Order Statistics ∴

$$F(x) = \int_{\theta}^{x} f(u)\,du = \int_{\theta}^{x} \frac{1}{1-\theta}\,du = \frac{1}{1-\theta}\int_{\theta}^{x} du$$

$$F(x) = \frac{1}{1-\theta}\left[u\right]_{\theta}^{x} = \frac{1}{1-\theta}\cdot(x-\theta) \quad , \quad \theta < x < 1$$

$$F_x(y_2) = \frac{y_2-\theta}{1-\theta} \quad , \theta < y_2 < 1$$

$$\therefore g(y_2) = 2\,(F_x(y_2))^{1}\cdot f_x(y_2) = 2\left[\frac{y_2-\theta}{1-\theta}\right]\cdot\frac{1}{1-\theta}$$

$$g(y_2) = \begin{cases} \dfrac{2(y_2-\theta)}{(1-\theta)^2} & , \theta < y_2 < 1 \\ 0 & , \text{o.w.} \end{cases}$$

∴ Order statistics ∴

Example:- let $f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , o.w \end{cases}$

and y_i, $i = 1, 2, 3$ are OS, find $g(y_2)$

Sol:- $n = 3$, $g(y_2)$ ingeneral

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \cdot \left(F_x(y_k)\right)^{k-1} \cdot \left(1 - F_x(y_k)\right)^{n-k} \cdot f_x(y_k)$$

$n = 3$, $k = 2$

$$\therefore g(y_2) = \frac{3!}{(2-1)!\,(3-2)!} \cdot \left(F_x(y_2)\right)^{2-1} \cdot \left(1 - F_x(y_2)\right)^{3-2} \cdot f_x(y_2)$$

$\because f(x) = 1 \Rightarrow f_x(y_2) = 1$

$$F(x) = \int_0^x f(u)\,du = \int_0^x du = u\Big]_0^x = x - 0 = x$$

$F_x(y_2) = y_2$ then $g(y_2) = 6\,(y_2)(1 - y_2) \cdot 1$

$$\Rightarrow g(y_2) = \begin{cases} 6\,(y_2)(1 - y_2) & , 0<y_2<1 \\ 0 & , o.w. \end{cases}$$

"(Order Statistics)"

Example:- let $f(x) = \begin{cases} 2(1-x) & , 0 < x < 1 \\ 0 & , \text{o.w.} \end{cases}$

and y_i, $i = 1, \dots, 5$ be OS, find $g(y)$ where $y = \max(x_i)$

Sol:- $n = 5$, find $y = \max(x_i) \rightarrow g(y_5)$

$$\Rightarrow g(y_n) = n\left(F_x(y_n)\right)^{n-1} \cdot f_x(y_n)$$

$$g(y_5) = 5\left(F_x(y_5)\right)^{4} \cdot f_x(y_5)$$

$$\rightarrow f(x) = 2(1-x) \quad \rightarrow f_x(y_5) = 2(1-y_5)$$

$$F(x) = \int_0^x f(u)\,du = \int_0^x 2(1-u)\,du = 2\int_0^x (1-u)\,du$$

$$F(x) = 2\left[u - \frac{u^2}{2}\right]_0^x = 2\left[x - \frac{x^2}{2}\right]$$

$$F_x(y_5) = 2\left(y_5 - \frac{y_5^2}{2}\right) \quad , 0 < y_5 < 1$$

$$g(y_5) = 5\left(2y_5 - y_5^2\right)^4 \cdot 2(1-y_5)$$