Chapter Three
Interpolation of Functions

I- Lagrange Interpolation

Let $f(x)$ be a real Continuous function on $[a,b]$, and we known it's values are $x_0, x_1, \dots, x_n$, then we can estimate values of $f(x)$,

x	x_0	x_1	x_2	$\dots$	x_n
$y=f(x)$	y_0	y_1	y_2	$\dots$	y_n

$$f(x^*) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{(x^* - x_i)}{(x_j - x_i)} \quad ; \; n = 0, 1, 2, \dots$$

x^* is the element which to find image to it.

Ex: By Lagrange formula, find the value of $f(\;)$ and $f(5)$, from the table ?

x	$\overset{x_0}{0}$	$\overset{x_1}{1}$	$\overset{x_2}{2}$	$\overset{x_3}{4}$
$f(x)$	1	1	2	5

Sol

1- $f(3)$, $x^* = 3$

$$f(x^*) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{(x^* - x_i)}{(x_j - x_i)}$$

$$f(3) = \sum_{j=0}^{3} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{3} \frac{(3 - x_i)}{(x_j - x_i)}$$

$$f(3) = f(x_0)\frac{(3-x_1)(3-x_2)(3-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(3-x_0)(3-x_2)(3-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ f(x_2)\frac{(3-x_0)(3-x_1)(3-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(3-x_0)(3-x_1)(3-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(3) = (1)\frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} + (1)\frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}$$

$$+ (2)\frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} + (5)\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}$$

$$f(3) = \frac{1}{4} - 1 + 3 + \frac{5}{4}$$

$$= 3.5 .$$

2. $f(5)$; $x^* = 5$

$$f(x^*) = f(5) = f(x_0)\frac{(5-x_1)(5-x_2)(5-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(5-x_0)(5-x_2)(5-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ f(x_2)\frac{(5-x_0)(5-x_1)(5-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(5-x_0)(5-x_1)(5-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(5) = (1)\frac{(5-1)(5-2)(5-4)}{(0-1)(0-2)(0-4)} + (1)\frac{(5-0)(5-2)(5-4)}{(1-0)(1-2)(1-4)}$$

$$+ (2)\frac{(5-0)(5-1)(5-4)}{(2-0)(2-1)(2-4)} + (5)\frac{(5-0)(5-1)(5-2)}{(4-0)(4-1)(4-2)}$$

$$f(5) = 6 .$$

Ex: Find the value of $f(2.1)$, where $x_0 = 1.1$, $x_1 = 1.7$, $x_2 = 3$ and $y_0 = 10.6$, $y_1 = 15.2$, $y_2 = 20.3$?

Sol

$$x^* = 2.1$$

$$f(x^*) = f(2.5) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x^* - x_i)}{(x_j - x_i)}$$

$$f(2.1) = f(x_0) \frac{(2.1 - x_1)(2.1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(2.1 - x_0)(2.1 - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{(2.1 - x_0)(2.1 - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$f(2.1) = (10.6) \frac{(2.1 - 1.7)(2.1 - 3)}{(1.1 - 1.7)(1.1 - 3)} + (15.2) \frac{(2.1 - 1.1)(2.1 - 3)}{(1.7 - 1.1)(1.7 - 3)} + (20.3) \frac{(2.1 - 1.1)(2.1 - 1.7)}{(3 - 1.1)(3 - 1.7)}$$

$$=$$

EXC: From the table

x	1	4	5
$y = f(x)$	16	88	0

Find ① $f(0)$?

② $f(2)$?

II Inverse of Lagrange interpolation

$$x^* = \sum_{j=0}^{n} x_j \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(y^* - y_i)}{(y_j - y_i)}$$

Ex Find the value of x^*, where $y^* = 2$ from the table

x	15	20	2
$y = f(x)$	1	3	5
	y_0	y_1	y_2

Sol

$$x^* = \sum_{j=0}^{n} x_j \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(y^* - y_i)}{(y_j - y_i)}$$

$$x^* = x_0 \frac{(2-y_1)(2-y_2)}{(y_0-y_1)(y_0-y_2)} + x_1 \frac{(2-y_0)(2-y_2)}{(y_1-y_0)(y_1-y_2)}$$

$$+ x_2 \frac{(2-y_0)(2-y_1)}{(y_2-y_0)(y_2-y_1)}$$

$$= (15) \frac{(2-3)(2-5)}{(1-3)(1-5)} + (20) \frac{(2-1)(2-5)}{(3-1)(3-5)}$$

$$+ (2) \frac{(2-1)(2-3)}{(5-1)(5-3)}$$

$$x^* = \frac{45}{8} + 15 - \frac{1}{4} = \frac{163}{8} = 20.375$$

Ex Use the following table to find x^* where $y^* = 0.2703$?

y	0.625	0.3443	0.2025
x	1.6	2.9	4.3

Sol

$$x^* = \sum_{j=0}^{2} x_j \prod_{\substack{i=0 \\ i\neq j}}^{2} \frac{(0.2703 - y_i)}{(y_j - y_i)}$$

$$= x_0 \frac{(0.2703-y_1)(0.2703-y_2)}{(y_0-y_1)(y_0-y_2)} + x_1 \frac{(0.2703-y_0)(0.2703-y_2)}{(y_1-y_0)(y_1-y_2)} + x_2 \frac{(0.2703-y_0)(0.2703-y_1)}{(y_2-y_0)(y_2-y_1)}$$

$$x^* = (1.6)\frac{(0.2703-0.3443)(0.2703-0.2025)}{(0.625-0.3443)(0.625-0.2025)} + (2.9)\frac{(0.2703-0.625)(0.2703-0.2025)}{(0.3443-0.625)(0.3443-0.2025)} + (4.3)\frac{(0.2703-0.625)(0.2703-0.3443)}{(0.2025-0.625)(0.2025-0.3443)}$$

$$x^* = 3.8658$$