

Theorem (1):

1. $\lim_{\theta \rightarrow \infty} \frac{\sin(\theta)}{(\theta)} = 0$
2. $\lim_{\theta \rightarrow \infty} \frac{\cos(\theta)}{(\theta)} = 0$
3. $\lim_{\theta \rightarrow \infty} \frac{1}{\theta} = 0$
4. $\lim_{\theta \rightarrow 0} \frac{1}{\theta} = \infty$

Theorem (2):

- 1- $\lim_{x \rightarrow \theta} \sin(x) = \sin(\theta)$
- 2- $\lim_{x \rightarrow \theta} \cos(x) = \cos(\theta)$
- 3- $\lim_{x \rightarrow 0} \frac{\sin(x)}{(x)} = 1$
- 4- $\lim_{\theta \rightarrow 0} \frac{\tan(\theta)}{(\theta)} = 0$

Theorem (3):

If f and g are two functions with $\lim_{x \rightarrow x_0} f(x) = L_1$ and

$\lim_{x \rightarrow x_0} g(x) = L_2$ then,

- 1- $\lim_{x \rightarrow x_0} k = k$, (k is any constant).
- 2- $\lim_{x \rightarrow x_0} x = x_0$
- 3- $\lim_{x \rightarrow x_0} c f(x) = cL_1$

$$4- \lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = L_1 \pm L_2$$

$$5- \lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = L_1 \cdot L_2$$

$$6- \lim_{x \rightarrow x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{L_1}{L_2}, \text{ such that } L_2 \neq 0.$$

Theorem (4):

$$1- \text{ if } \lim_{x \rightarrow x_0} f(x) = L, \text{ then}$$

$$\lim_{x \rightarrow x_0} [f(x)]^n = [\lim_{x \rightarrow x_0} f(x)]^n = [L]^n$$

$$2- \lim_{x \rightarrow x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow x_0} f(x)} = \sqrt[n]{L}, L \geq 0, n \text{ is even.}$$

Theorem (5):

If $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n$
is any polynomial function, then

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) = a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + a_4x_0^4 + \dots + a_nx_0^n$$

Examples:

1- Prove that $\lim_{h \rightarrow 0} \frac{2 \sin(h) \cdot \cos(h)}{h} = 2$

Solution:

To prove $\lim_{h \rightarrow 0} \frac{2 \sin(h) \cdot \cos(h)}{h} = 2$

From

$$\sin(2h) = 2 \sin(h) \cdot \cos(h)$$

So $\lim_{h \rightarrow 0} \frac{2 \sin(h) \cdot \cos(h)}{h} = \lim_{h \rightarrow 0} \frac{\sin(2h)}{h}$

$$\lim_{h \rightarrow 0} \frac{\sin(2h)}{h} = 2 \lim_{h \rightarrow 0} \frac{\sin(2h)}{2h}$$

By using theorem (2),

$$\lim_{h \rightarrow 0} \frac{2 \sin(h) \cdot \cos(h)}{h} = 2$$

2- Find $\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x + \cos(x)}$

Solution:

$$\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x + \cos(x)} = \lim_{x \rightarrow \infty} \frac{\frac{x}{x} + \frac{\sin(x)}{x}}{\frac{x}{x} + \frac{\cos(x)}{x}}$$

بالضرب

$$\frac{2}{2} \times$$

بالقسمة

على x

$$= \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin(x)}{x}}{1 + \frac{\cos(x)}{x}}$$

_by using theorem (1).

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{\sin(x)}{x}}{1 + \frac{\cos(x)}{x}} = \lim_{x \rightarrow \infty} \frac{1 + 0}{1 + 0} = 1$$

3- Find $\lim_{x \rightarrow 0} \frac{\sin^2 x}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \frac{\sin(x) \cdot \sin(x)}{x}$$

$$= \lim_{x \rightarrow 0} \left[\sin(x) \cdot \frac{\sin(x)}{x} \right]$$

$$= \lim_{x \rightarrow 0} \sin(x) \cdot \lim_{x \rightarrow 0} \frac{\sin(x)}{x}$$

By using theorem (2)

$$= \lim_{x \rightarrow 0} \sin(x) \cdot 1 = 0$$

4- Find $\lim_{x \rightarrow 0} \frac{\sin(4x)}{x}$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 4x}{x} &= \lim_{x \rightarrow 0} \frac{4 \sin(4x)}{4x} \\ &= 4 \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x} \end{aligned}$$

بالضرب

$$\frac{4}{4} \times$$

By using theorem (2).

$$= 4 \cdot (1) = 4$$

5- Find $\lim_{x \rightarrow \frac{\pi}{2}} \sin(x) = 0$

Solution:

$$\lim_{x \rightarrow \frac{\pi}{2}} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

بالتعويض المباشر

6- Find $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(5x)}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{\frac{(3x) \cdot \sin(3x)}{3x}}{\frac{(5x) \cdot \sin(5x)}{5x}}$$

بالضرب مرة $\frac{3x}{3x} \times$

ومرة ثانية $\frac{5x}{5x} \times$

$$= \frac{\lim_{x \rightarrow 0} (3x) \cdot \lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \right)}{\lim_{x \rightarrow 0} (5x) \cdot \lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \right)}$$

By using theorem (2).

$$= \frac{\lim_{x \rightarrow 0} (3x) \cdot 1}{\lim_{x \rightarrow 0} (5x) \cdot 1}$$

$$= \lim_{x \rightarrow 0} \left(\frac{3x}{5x} \right) = \lim_{x \rightarrow 0} \left(\frac{3}{5} \right) = \frac{3}{5}$$

Remark:

1- If the value of $f(x)$ approaches to L_1 as x Approaches to x_0 from the right side we write
 $\lim_{x \rightarrow x_0^+} f(x) = L_1.$

2- If the value of $f(x)$ approaches to L_2 as x Approaches to x_0 from the left side we write
 $\lim_{x \rightarrow x_0^-} f(x) = L_2.$

3- If the limit from the left side equals the limit from the right side (i.e if $L_1 = L_2 = K$)

Then $\lim_{x \rightarrow x_0} f(x)$ exist and $\lim_{x \rightarrow x_0} f(x) = k$

But if $L_1 \neq L_2$,Then the $\lim_{x \rightarrow x_0} f(x)$ does not exist.

Example:

Find $\lim_{x \rightarrow 3} f(x)$ if it exists where

$$f(x) = \begin{cases} 4+x & ,x \geq 3 \\ x-4 & ,x < 3 \end{cases}$$

Solution:

Right side

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (4 + x) = 4 + 3 = 7 = L_1$$

Left side

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x - 4) = 3 - 4 = -1 = L_2$$

Now, since

$\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$, then $\lim_{x \rightarrow 3} f(x)$ does not exist.

Example:

find $\lim_{x \rightarrow 2} f(x)$ if it exists

$$f(x) = \begin{cases} x+4 & \text{if } x \geq 2 \\ x^2+2 & \text{if } x < 2 \end{cases}$$

Solution:

Right side

$$\lim_{2^+} f(x) = \lim_{2^+} (x + 4) = 2 + 4 = 6$$

Left side

$$\lim_{2^-} f(x) = \lim_{2^-} x^2 + 2 = 2^2 + 2 = 6$$

Since $\lim_{2^+} f(x) = \lim_{2^-} f(x) = 6$

Thus $\lim_{x \rightarrow 2} f(x) = 6$