7/1/2026 محاضرة الاحصاء

"Efficiency"

Def:- if $v(\hat{\theta})$ is equal to CRLB, then $\hat{\theta}$ is called efficient estimator for θ.

i.e.

$$eff = \frac{CRLB}{v(\hat{\theta})}$$

where $CRLB = \frac{1}{I} = I^{-1}$

Cramer- Rao Lower Bound.

where I is called by Fisher the amount of information about θ contained in observation x.

$$I = nE\left[\frac{\partial(\ln(f(x,\theta)))}{\partial\theta}\right]^2$$

or $$I = -nE\left[\frac{\partial^2(\ln(f(x,\theta)))}{\partial\theta^2}\right]$$

and $v(\hat{\theta}) \geq I^{-1}$

Example:- let $X_i \sim P(\theta)$. $i = 1,2,\dots,n$. Find CRLB and show that $\hat{\theta} = \bar{x}$ is efficient estimator for θ.

Sol:-

$\because X_i \sim P(\theta)$

$$\Rightarrow f(x,\theta) = \frac{e^{-\theta}\cdot\theta^x}{x!}$$

$$\Rightarrow \ln f(x,\theta) = \ln\left[\frac{e^{-\theta}\cdot\theta^x}{x!}\right] = -\theta + x\ln(\theta) - \ln(x!)$$

$$\frac{\partial(\ln(f(x,\theta))}{\partial\theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$\rightarrow nE\left[\frac{\partial(\ln(f(x,\theta))}{\partial\theta}\right]^2 = nE\left[\frac{x-\theta}{\theta}\right]^2$$

$$= \frac{n}{\theta^2}E[x-\theta]^2 = \frac{n}{\theta^2}V(x) = \frac{n}{\theta^2}\cdot\theta$$

$$= \frac{n}{\theta}$$

$$\Rightarrow CRLB = \frac{1}{I} = \frac{\theta}{n}$$

$\rightarrow$ Now for efficiency we have that :-

$$eff(\hat{\theta}) = \frac{CRLB}{V(\hat{\theta})}$$

where $V(\hat{\theta}) = V(\bar{x}) = \frac{\sum_{i=1}^{n} V(x_i)}{n^2} = \frac{\sum_{i=1}^{n}\theta}{n^2} = \frac{n\theta}{n^2}$

$$\Rightarrow V(\hat{\theta}) = \frac{\theta}{n}$$

$$\therefore eff(\hat{\theta}) = \frac{\theta/n}{\theta/n} = 1$$

$\rightarrow$ So $\hat{\theta} = \bar{x}$ is efficient estimator for θ.

H.W :- let $X_i \sim Exp(\theta)$. $i=1,2,\dots,n$. Find CRLB and show that $\hat{\theta} = \bar{x}$ is efficient estimator for θ.

"Efficiency for more than one estimator"

Def:- if we have $\hat{\theta}_1$ and $\hat{\theta}_2$ are two unbiased estimators for θ. then efficient estimator is one with less variance.

$$eff = \frac{v(\hat{\theta}_1)}{v(\hat{\theta}_2)}$$

* اذا كانت النسبة أقل من 1 فان المقدر الاول هو الاكفأ والعكس صحيح.

Ex:- let X_i, $i = 1,2,3$. be indep. r.v.s with mean θ and variance σ^2. Find the efficient estimator between

$$\hat{\theta}_1 = \frac{(x_1+2x_2+3x_3)}{6} \quad \text{and} \quad \hat{\theta}_2 = \bar{x}$$

Sol:-

1. $$E(\hat{\theta}_1) = E\left[\frac{(x_1+2x_2+3x_3)}{6}\right]$$

$$= \frac{E(x_1)+2E(x_2)+3E(x_3)}{6} = \frac{\theta+2\theta+3\theta}{6} = \frac{6\theta}{6} = \theta$$

$$\Rightarrow E(\hat{\theta}_2) = E(\bar{x}) = \theta$$

So they are unbiased estimators.

2. $$v(\hat{\theta}_1) = v\left[\frac{(x_1+2x_2+3x_3)}{6}\right] = \frac{1}{36}\left[v(x_1)+4v(x_2)+9v(x_3)\right]$$

$$= \frac{\sigma^2+4\sigma^2+9\sigma^2}{36} = \frac{7}{18}\sigma^2$$

$v(\hat{\theta}_2) = v(\bar{x}) = \frac{\sigma^2}{3}$. So since $v(\hat{\theta}_1) > v(\hat{\theta}_2)$

Then $\hat{\theta}_2 = \bar{x}$ is the efficient estimator for θ.